Algebra, geometry, and combinatorics of link homology

organized by
Nicolle Gonzalez, Eugene Gorsky, Matthew Hogancamp, Oscar Kivinen, and Alexei Oblomkov

Workshop Summary

Introduction

In this section, introduce the workshop and provide some background information about its objectives, participants, and focus. Mention the workshop's official webpage as the source of information.

This workshop is devoted to the recent developments in the intersection of Khovanov-Rozansky homology, affine Springer fibers, Hilbert schemes and link homology, and combinatorics. The main topics for the workshop are:

1. Computing Khovanov-Rozansky link homology for large families of links: Khovanov and Rozansky defined link homology categorifying the HOMFLY-PT link polynomial. This homology was computed explicitly for all torus links, but remains unknown for cables of torus knots and other algebraic links. Conjectures of Oblomkov, Rasmussen and Shende and their extensions give algebro-geometric description for this homology in terms of Hilbert schemes of points on singular curves, affine Springer fibers, Hitchin's integrable system, orbital integrals in algebraic number theory and even Coulomb branches of gauge theories. We plan to revisit these conjectures, relate all these various descriptions and attempt to match them with the knot homology side. Link homology and Hilbert schemes on the plane: another set of conjectures of Gorsky, Negut, Rasmussen, Oblomkov, Rozansky and others relates Khovanov-Rozansky homology to the Hilbert scheme of points on the plane. This variety plays a central role in modern geometric representation theory. We plan to find the analogues of geometric structures on Hilbert schemes (coordinate ring, Poisson bracket, vector bundles and their sections etc) in link homology.

2. Combinatorics of Shuffle Conjecture and beyond: In algebraic combinatorics, the Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel and Ulyanov spurred a lot of activity and interest. This conjecture gives a combinatorial expression for matrix elements of a certain operator on symmetric functions, which turns out to be related to the action of the full twist braid in link homology. While the Shuffle conjecture was recently proven by Carlsson and Mellit, many of its cousins are closely related to other computations in link homology and remain open.

3.Categorical skein theory: The skein theory which underlies the HOMFLY-PT polynomial is closely related to the theory of symmetric functions. Various topological operations such as adding a meridian or a full twist can be interpreted as certain operators on symmetric functions. A general, and more abstract direction to approach the above problems is related to the categorification of the skeins of the annulus and of the torus, and their relation to the categorifications of the Heisenberg and elliptic Hall algebras, as well as the categorification of the Carlsson-Mellit algebra which appeared in their proof of the Shuffle Conjecture.

Workshop Overview

Provide a brief overview of the workshop's schedule, topics covered, and any notable events that took place during the workshop. Highlight key discussions, presentations, and sessions.

Monday, July 31, 2023

9:00 am Welcome and Introductions

Eugene Gorsky gave an overview talk on algebra-geometric models for Khovanov-Rozansky homology which include braid varieties, affine Springer fibers and Hilbert schemes of points on the plane. **Paul Wedrich** gave an overview talk about topological invariance and functoriality of various link homologies, including HOMFLY homology and colored $\mathfrak{gl}_N$ homologies.

2:00 pm Problem Session: the participants compiled a very long list of over 35 problems with the help of José Simental and Tonie Scroggin. The problem list is typed and will be posted at the AIM webpage.

Happy Hour

Tuesday, August 1, 2023 9:00 am **Pavel Galashin** defined positroid varieties and talked about their relation with links in a thickened torus and the elliptic Hall algebra. **Catharina Stroppel** talked about Richardson varieties in Grassmannians, their Deodhar-type stratifications and homology.

2:00 pm Working Groups

Happy Hour and Reception

Wednesday, August 2, 2023 9:00 am **Jim Haglund** talked about the combinatorics of q, t-Catalan numbers, diagonal coinvariant rings and their super-analogues. **Mikhail Mazin** talked about affine Springer fibers and compactified Jacobians for (unibranched) plane curve singularities, and the combinatorics of their cell decompositions.

2:00 pm Working Groups

Happy Hour

Thursday, August 3, 2023 9:00 am **José Simental** defined braid varieties and talked about their geometry and compactifications by briack manifolds. **Alexei Oblomkov** talked about his work with Lev Rosansky relating link homology, matrix factorizations and Hilbert schemes of points on the plane.

2:00 pm Working Groups

Happy Hour

Friday, August 4, 2023 9:00 am **Joshua Wang** talked about representation varieties for fundamental groups of link complements, and their connection to link homology. **Mikhail Khovanov** talked about the foundations of link homology via the foam evaluation formula.

2:00 pm Working Groups

Happy Hour

Collaborative Activities

If there were any collaborative activities, group exercises, or brainstorming sessions, describe their outcomes and their significance in advancing the workshop's objectives.

Working Groups.

1. Invariants of algebraic knots in lens spaces(lead: Peter Samuelson).

Given a polynomial $f(x, y)$ in two variables, the solution set to the equation $f = 0$ is a (complex, usually singular) curve in $\mathbb{C}^2$. Intersecting this solution set with a small ball around a singular point gives a knot or link in S^3. About 10 years ago, Oblomkov and Shende conjectured (and Maulik proved) a precise formula relating algebro-geometric invariants of the curve to topological invariants of the knot/link. This room's problem was to generalize this relationship by replacing $\mathbb{C}^2$ with a singular surface. The main question the room discussed was where such invariants will live, and discussions seemed to indicate that if the surface is $\mathbb{C}^2/(\mathbb{Z}/p)$, then there is a p-dimensional vector space where invariants on both sides should live. The room intends to continue working on this problem, and the next step is to compute some examples that were too long to finish during the week.

2. $SU(N)$**-character varieties and braid varieties** (lead: Josh Wang).

The braid variety associated to a positive braid is a smooth variety whose cohomology recovers the lowest a-degree part of triply-graded Khovanov–Rozansky homology of the braid closure. Given a particular positive braid word representing the positive braid, a point in the variety may be thought of as a sequence of flags in $\mathbb{C}^n$ subject to some constraints. Here n is the number of strands of the braid.

Associated to a link is a variety of $SU(N)$ representations of the fundamental group of the complement of the link. Here, only the representations sending meridians to a particular conjugacy class are considered. The cohomology of this variety is known to match the $sl(N)$ Khovanov–Rozansky homology of the link in some cases, e.g the torus link $T(2, m)$. Given a diagram of the link, a point in the representation variety may be thought of a configuration of lines in $\mathbb{C}^N$, one for each arc of the diagram, subject to constraints.

A group made an effort to compare, contrast, and potentially relate these constructions. At the surface, they are quite different. For example, braid varieties are smooth complex varieties while $SU(N)$ representation varieties are potentially singular compact real varieties. They are also defined for different classes of topological objects and are related to different link homology theories. However, the analogues of these varieties for planar braid-like webs appear much more closely related. The group also attempted to define braid varieties in the presence of negative crossings and to define analogues of representation varieties for triply-graded Khovanov–Rozansky homology, though the proposals so far remain unsatisfactory and leave much room for further investigation.

3. Combinatorics of super version of the double coinvariants and knot homology (lead: Jim Haglund).

Zabrocki has conjectured that the trigraded character of the super diagonal coinvariant ring SDR_n is given by a certain sum involving Delta operators, which generalize the nabla operator. This sum was conjectured by Haglund, Remmel and Wilson to also have a combinatorial expression in terms of weighted parking functions, which was recently proved by Dâ€™Adderio and Mellit using the Dyck path algebra, and also (independently) by Blasiak, Haiman, Morse, Pun and Seelinger, using the Elliptic Hall algebra). The project we (primarily Josh Swanson, Andy Wilson and Jim Haglund) were looking at during the AIM workshop is how to prove Zabrockiâ€™s conjecture, and also how to find an extension of the Carlsson-Oblomkov (CO) basis for diagonal coinvariants DR_n to SDR_n. (Haglund and Sergel have a paper giving a candidate monomial basis for SDR_n, which is based on a â€œschedules formulaâ€ for the Hilbert series of the sum involving Delta operators mentioned above. This formula reduces to the schedules formula for the Hilbert series of ∇e_n of Haglund and Loehr, which has an important interpretation in the CO basis. Hence the combinatorics underlying the CO basis extends nicely to the SDR_n case.) During the workshop, co-organizer Alexei Oblomkov presented a possible way of extending the geometric setup from his proof with Carlsson of the CO basis result to SDR_n. His idea is to replace GL_n/B in their proof by GL_n/U, where B is a Borel subgroup and U a nilpotent subgroup. In one afternoon session he (with some help from us, and Pavel Galashin) tried to compute the n = 2 case of his GL_n/U setup; in order to possibly lead to a monomial basis for SDR_n, it would need to be of dimension 4. We reduced the calculation to computing the cohomology of a certain expression. We were not able to complete it during the workshop, but a few days after Alexei emailed us to say he found a way to compute this cohomology, and it did have dimension 4. So, this approach is promising. Alexei is hoping to do the $n = 3$ case; the target dimension is 28 (the dimension of SDR_3).

With regards to Zabrockiâ€™s conjecture, workshop co-organizer Matt Hogancamp presented a pos- sible approach which involved generalizing Haimanâ€™s proof of the ∇e_n theorem. Here the Hilbert scheme of n points in $\mathbb{C}^2$ is the central object; Matt suggested that we look at $\mathrm{Hom}(\mathcal{P}, \Delta^*(T)|_{\mathrm{Hilb}_n(\mathbb{C}^2,0)})$, where $\mathcal{P}$ is the Procesi bundle, and $\Delta^*(T)$ the exterior algebra of the tautological bundle. Josh, Andy and I do not currently know enough about the Hilbert scheme to be able to compute this by ourselves, but the idea seems really interesting, and we are currently trying to compute this when $n = 2$, with $\mathbb{C}^2$ replaced by $\mathbb{C}^1$. (Note: the special case of Zabrockiâ€™s conjecture with one set of bosonic variables set to 0 (where SDR_n becomes SR_n, the super coinvariant ring) is also open, and has itself become something of a famous problem in algebraic combinatorics. Recently Rhoades and Wilson proved the Hilbert series case of this, but nobody has proved a monomial basis for it yet.) In summary, we have intriguing new geometric approaches to both problems involving SDR_n, which would extend central results in this area about DR_n.

4. Closures of categorified projectors in the annulus.(lead: Paul Wedrich).

There are two conceptually distinct strategies for upgrading link homology theories to a colored setting. Both strategies start by taking parallel cables of framed link components, but then they diverge. The first strategy realizes the colored link homology as a subspace of the link homology of the cable, while the second strategy computes homology after inserting a categorified projector into the cable. The first strategy, unlike the second, produces answers that are supported in finitely many homological degrees, but otherwise not much is know about their relationship.

At AIM we set out to compare both types of colored homologies in an annular model situation, namely the dg horizontal trace of type A Soergel bimodules, which was computed by Gorsky-Hogancamp-Wedrich. We discussed in which sense this gadget categorifies the ring of (q, t)-symmetric functions and how the first strategy of colored link homology gives rise to a generating set of “Schur objects”. Implementing the second strategy, on the other hand, one is led to compute the Schur-expansions of annular closures of categorified central projectors that were introduced by Elias-Hogancamp. We did this for small examples and compared the answers with the combinatorics of modified Macdonald polynomials. We will continue our discussion through online meetings.

5. Parabolic braid varieties(lead: Jose Simental)

The goal of the group was to define and study a parabolic analog of braid varieties satisfying many of the usual properties of braid varieties. For example:

-Parabolic braid varieties should be smooth and affine.

-Parabolic Richardson varieties should arise as special cases of parabolic braid varieties.

-Parabolic braid varieties should admit Deodhar-type decompositions and, in the best of cases, a cluster structure.

During the week, we gave a definition of parabolic braid varieties and verified that, using this definition, every parabolic Richarson variety is indeed a parabolic braid variety. We also verified that parabolic braid varieties are affine and discussed ideas towards smoothness. In the coming months, we plan to keep working on these and analyze Deodhar type decompositions using two sets of ideas: via a weave-like calculus, and via the Bruhat decomposition of the general linear group. We also plan to leverage the known cluster structures on usual braid varieties in order to construct cluster structures on parabolic ones.

6. Cabling of torus knots and Coxeter (monotone) links (lead: Matt Hogancamp)

The main goal of this group was to modify known techniques for computing HHH for torus links, and apply them to larger classes of links, for instance monotone links as taught to us by Pavel Galashin. We spent one afternoon reviewing the Elias-Hogancamp recursion, involving categorified Young symmetrizers. On the afternoon of the final day we successfully computed HHH of the (2,13) cable of the trefoil. We hope to further understand the techniques and various recursions introduced by Elias-Hogancamp and Gorsky-Mazin-Oblomkov in a more general setting. Several members of this group are continuing to meet over Zoom, and are in the process of writing a paper on the results of this activity.

7. Computation of the Ext groups for some class of Soergel bimodules (lead: Cailan Li).

We initially worked on computing the algebra structure of $\mathrm{Ext}^{\bullet;\bullet}_{R^e}(B_{w_0}, B_{w_0})$ where $R^e = R \otimes R$ and B_{w_0} is the indecomposable Soergel Bimodule corresponding to the longest element of the coxeter group W where $R = \mathrm{Sym}(V^*)$ where V is the geometric realization of W. In general we found this to be $\Lambda^\bullet(U) \otimes (R \otimes_{R^{S_n}} R)$ for a certain graded vector space U. We then worked on understanding the category $A_{\mathrm{Ext}} - \mathrm{mod}$ where $A_{\mathrm{Ext}} := \mathrm{Ext}^{\bullet;\bullet}_{R^e}(\oplus_{w\in W} B_w, \oplus_{w\in W} B_w) \otimes_R \mathrm{C}$. Replacing Ext with Hom we recover Category O when W is a Weyl group so this is a natural generalization.

Diagrammatics for Ext groups between Soergel Bimodules were developed for $W = S_2$ (the A_1 case) by Makisumi and in dihedral type by Li. Using these diagrammatics, we were able to compute the quiver for A_{Ext} when in type A_1 and make some progress in dihedral type. In type A_1 we have that $R \cong \mathbb{C}[x]$, and define

$$A^t_{\mathrm{Ext}} := \mathrm{Ext}^{\bullet;\bullet}_{R^e}(\oplus_{w\in W} B_w, \oplus_{w\in W} B_w) \otimes_R \mathrm{C}_t$$

where $\mathrm{C}_t = \mathrm{C}$ is the R module where x acts by the scalar $t \in \mathbb{C}$ and note that setting $t = 0$ recovers A_{Ext} as defined above. For $t \neq 0$ we showed that $A^t_{\mathrm{Ext}} - \mathrm{mod}$ was highest weight, while at $t = 0$ it wasn't highest weight (or rather does not have a triangular basis in the sense of Brundan). Nevertheless, Brundan's theory allows us to construct pseudo-standard and pseudo-costandard modules (these would be standard and costandard modules had $A^t_{\mathrm{Ext}} - \mathrm{mod}$ been highest weight) and we examined the socle and head of these modules.

We plan to prove a double centralizer theorem with $\mathrm{Ext}^{\bullet,\bullet}_{R^e}(B_{w_0}, B_{w_0})$ and A_{Ext} which would be a natural generalization of Soergel's Struktursatz. We also want to develop an analogue of Brudan's theory of triangular bases suitable for the algebra A_{Ext}.

Conclusion

The workshop was overwhelmingly a success. Despite the difficulties of bringing together people from so many different backgrounds, all participants were engaged in conversations with many new collaborations and ideas arising from the process. There were many interesting questions proposed by the participants which we were unable to address, but hope people are able to discuss and solve later. Many of the participants were also early career and this workshop served as an introduction to those new to the area and opened their eyes to the myriad of approaches being applied to solve related questions. It is our hope that this younger generation will continue working in the directions proposed during this meeting and expand our current knowledge further. We will undoubtedly hold future related meetings in the future where solutions to many of the problems discussed this time will be presented.

Acknowledgments

We are thankful to AIM for making this possible. We also thank the participants and speakers for their interest, time, and wonderful questions which made this workshop a success.

Bibliography

[aimathworkshop] American Institute of Mathematics. (Year). *Workshop Title*. Retrieved from https://aimath.org/pastworkshops/agclinkhom.html