Albertson conjecture and related problems

organized by
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Workshop Summary

This workshop was devoted to Albertson's conjecture and to other exciting open problems related to crossing numbers, graph embeddings, and colorings. We invited a diverse group of 22 excellent participants from the US, Canada, Hungary, Mexico, and the Philippines, at different stages of their careers. They compiled a list of more than a dozen open problems in the field, studied them in several discussion groups, and made important partial progress toward their solution. The work initiated during the workshop continues and is likely to lead to further results, discoveries, and papers.

Towards Albertson's conjecture

Albertson's conjecture states that every graph G with chromatic number $\chi(G) \geq r$ has crossing number at least that of the complete graph on r vertices. In notation,

$$\chi(G) \geq r \implies \mathrm{cr}(G) \geq \mathrm{cr}(K_r).$$

The workshop started with a survey lecture by Jacob Fox, who summarized the main results related to the problem and the known techniques used in the literature. This section summarizes our discussion and progress on Albertson's conjecture during the workshop.

It is known that the conjecture is true [ACF09, BT10, Ackerman19] for graphs G with $n \leq 18$ or $n \geq 3.03r$ where $n = |V(G)|$. All previous works follow the same strategy: First, we notice that it is sufficient to prove the conjecture for r-critical graphs. Next, we establish a lower bound on the edge-density of r-critical graphs and apply the Crossing Lemma of Leighton [Leighton] and Ajtai et al. [Ajtai] to conclude that such graphs have many crossings. Lastly, we compare the number of crossings guaranteed in the previous step with $H(r) = \frac{1}{4}\lfloor\frac{r}{2}\rfloor\lfloor\frac{r-1}{2}\rfloor\lfloor\frac{r-2}{2}\rfloor\lfloor\frac{r-3}{2}\rfloor$, which is known be an upper bound $\mathrm{cr}(K_r)$. (In fact, Harary and Hill famously conjectured $H(r) = \mathrm{cr}(K_r)$.)

During this workshop, we concluded that both bounds ($n \leq 18$ or $n \geq 3.03r$) can be improved following a similar strategy, using the recently obtained better constants in the crossing number inequality [BK24]. We noticed that, to lower bound the edge density, all previous works [ACF09, BT10, Ackerman19] only used the elementary fact that r-critical graphs have minimum degree $r - 1$. During our discussion, we speculated that a more sophisticated argument could be made at this step in order to obtain a further gain on the ranges where Albertson's conjecture holds.

In the paper of Barát and Tóth [BT10], it was also proved that Albertson's conjecture holds for $n < r + 4$. During his talk, Jacob Fox suggested that this condition could be

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improved to $n < (1+c)r$ for some positive constant c and large enough r.[1] The suggested approach is closely related to the following conjecture of Lescure and Meyniel, closely related to the famous Hadwiger conjecture, one of the central unsolved questions in graph coloring theory: Every graph with chromatic number r contains a weak immersion of the complete graph K_r. A *weak immersion of* K_r is a set of r vertices and a collection of $\binom{r}{2}$ edge-disjoint paths connecting them. (Here, a path between two vertices is allowed to pass through some other vertices.)

The relation between weak immersions and crossing numbers is as follows: if there is a weak immersion of K_r in a graph G, for any drawing of G, we may concatenate the paths between the r immersed vertices to form a curve, hence heuristically creating a drawing of K_r. However, we may need to perturb the curves around the additional vertices along the paths (to avoid immersed vertices along the connecting curves). However, the number of extra crossings caused by such perturbations can be controlled. Finally, in view of the structural characterization of r-critical graphs (say, by Gallai's lemma [Gallai63]), it can be conjectured that using these methods both the Lescure–Meyniel conjecture and Albertson's conjecture can be verified in the range $n \leq 2r - o(r)$.

Some related problems

In this section, we list some other questions related to Albertson's conjecture, and summarize the results concerning them achieved by the participants during the workshop.

First, we mention a question that often appears as a subproblem in papers about crossing numbers: how continuous is this parameter? More precisely, how much can it change by the addition of a single edge (http://aimpl.org/albertson/1/)?

Problem (Jacob Fox)**.** Does there exist a constant $c < 1$ such that any graph G with m edges and any two non-adjacent vertices u, v in G satisfy $\mathrm{cr}(G + uv) \leq \mathrm{cr}(G) + c \cdot m$?

János Pach and Géza Tóth showed that the trivial upper bound $\mathrm{cr}(G) + m$ can be reduced by 4. László Székely and Éva Czabarka noted that, since $K_{3,3}$ minus an edge is planar, the desired statement is not true with $c < 1/8$, and in the above improvement 4 cannot be replaced by 8.

Jeck Lim, Dylan King, and László Székely showed that $c = 3/4$ works for a different kind of crossing number, the *spheric linear crossing number*, $\mathrm{slcr}(G)$. That is, we have

$$\mathrm{slcr}(G + uv) \leq \mathrm{slcr}(G) + \frac{3}{4}m.$$

In a spheric linear drawing of G, the vertices are placed on a sphere and the edges are pieces of great circles. Once the positions of the vertices have been fixed, for each edge uv, they randomly select which of the two great circle pieces between u and v will represent uv. For every graph G, its spheric linear crossing number is sandwiched between the classical crossing number and the rectilinear crossing number of G. It is unclear whether a similar argument can be made for classical crossing numbers. The same group (plus Shira Zerbib) also made the following conjecture that would imply a positive answer to Fox's problem: Let G be a simple topological graph drawn in the plane and G' be its planarization. Then $\mathrm{diam}(D(G')) \leq c|E(G)|$ for some absolute constant $c < 1$. Here, a "planarization" is obtained by adding vertices in every crossing point. The dual $D(G)$ of a plane graph G is the graph whose vertices are the faces of G and whose edges are pairs of faces that share a side.

[1] In fact, this has already been achieved in a recent unpublished manuscript by Fox, Pach, and Suk.

Another group of participants worked on problems related to chromatic numbers, the other main ingredient in Albertson's conjecture (http://aimpl.org/albertson/8/). We define the *generalized Kneser graph* $KG_s(n, k)$, as follows: its vertex set consists all size-k subsets $K \subset [n]$ with $s \leq |i - j| \leq n - s$ ($\forall i, j \in K$), where two vertices, K and K', are connected by an edge if and only they are disjoint.

Problem (Shira Zerbib)**.** Is it true that for $n > sk$, the chromatic number $\chi(KG_s(n, k)) = n - sk + s$?

Shira Zerbib noted that it follows by earlier results [Schrijver78, Jonsson12, Chen15] that the only interesting case of this problem is when $s = 3$. Jeck Lim, Cosmin Pohoata, and Gábor Tardos concentrated to this problem, using the result of Dol'nikov [Dolnikov88], according to which, for any collection of subsets F of $[n]$, we have $\chi(KG(F)) \geq cd_2(F)$, where $KG(F)$ denotes the Kneser graph on the vertex set F (whose edges correspond to disjoint pairs of sets in F). The parameter $cd_2(F)$ is the so-called *2-colorability defect* of F (which comes from a different coloring problem). Using this approach, one can obtain an alternative proof of Lovász' theorem which implies Kneser's conjecture. This is the main motivation for Zerbib's problem.

Naturally, we studied the parameter $cd_2(F_s)$, where F_s consists of every size-k subset $K \subset [n]$ with $s \leq |i - j| \leq n - s$ for all $i, j \in K$. We realized that $cd_2(F_s) = n - 2sk + s$, for each $s \geq 2$, so that proving the inequality $\chi(KG_s(n, k)) \geq n - sk + s$ is equivalent to improving the $-2sk$ term to $-sk$. If s is even, we achieved this by appropriately refining the proof of Dol'nikov's theorem. This recovers the result of Chen [Chen15] concerning the original problem. It is unclear whether this approach would completely settle the problem for $s = 3$, but there is still hope in this direction.

The following problem on planar-critical graphs also received attention (http://aimpl.org/albertson/1 A graph is said to be k-planar if it can be drawn with at most k crossings per edge. A graph is said to be *k-planar-critical* if it is not k-planar but the removal of any edge makes it k-planar.

Problem (Géza Tóth)**.** Does there exist a constant k such that every 1-planar-critical graph is k-planar?

As noted by Géza Tóth, this statement implies that every m-planar-critical graph is k-planar for some $k = k(m)$. Towards a lower bound on the value of k from the statement, the participants were not able to rule out the possibility that all 1-planar-critical graphs are 2-planar. Regarding the upper bound, we obtained a positive result in the case when the minimum degree $\delta(G) \geq 3$ and the maximum degree $\Delta(G)$ of G is bounded from above by a constant. In particular, we get $k = O(\Delta(G))$. To this end, we leveraged a technique of Richter and Thomassen [RT93], and a recent result of Dross [Dross20] stating that every 1-planar graph with $\delta(G) \geq 3$ contains a cycle of length at most 198. As a next step in our investigation, we would like to see if the technique in [RT93] gives a positive result when the condition on the maximum degree is dropped. In particular, we would like to know if the above results extend to 1-planar graphs. We observed that for any k there is non-k-planar graph so that there is subset of the edges, which is a positive fraction of all the edges, so that if we remove any one of them, the resulting graph is planar. If we can increase this subset to all edges, (with 1-planar in place of planar) it would give a "no" answer to the above question.

Finally, some participants made progress on the following problem about crossings in geometric graphs that are "not far" from being planar (http://aimpl.org/albertson/14/). For a point set P in the plane containing no collinear quadruple, define a geometric graph G_P on the vertex set P by adding the edges uv and vw, for every collinear triple (u, v, w), in this consecutive order along the line.

Problem (József Solymosi)**.** What is the minimum number such that for every set P of n points with this many collinear triples, but no collinear quadruple, the graph G_P contains a self-intersecting path of length 3?

A group of participants, including Hisatsuga, Pohoata, Solymosi, Suk, and Tardos, discussed the literature [PPGT04] behind the more general question about self-intersecting 3-paths in geometric graphs that do not necessarily come from collinear triples. They managed to adapt the construction from [PPGT04] to prove that there exist configurations P of n points with $\Omega(n(\log n)^{1/2})$ collinear triples, for which G_P has no self-intersecting 3-paths. It remains an interesting open question to decide whether the bound $O(n(\log n)^{1/2})$ is best possible.

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