From ₂ to infinity

organized by
James Cummings, Itay Neeman, and Dima Sinapova

Workshop Summary

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A central theme in modern set theory is investigating ZFC-constraints on infinite cardinals. The main tools are forcing techniques, such as Prikry type forcing and proper forcing, and combinatorial properties such as the tree property and stationary reflection. The workshop brought together researchers involved with these topics, so people can learn about the new developments from each other and then delve into problems. The participants were a mix of young researchers and more established experts.

The structure of the workshop.

Every morning there were two talks, which varied from tutorial style to discussion of key new developments.

One of the discussed topics was recent advances in forcing regarding singular combinatorics, such as Gitik's diagonal overlapping extenders forcing, iterated ultrapowers to read Prikry sequences, and abstract frameworks for iterating Prikry type forcing. On the first day, Cummings went over Prikry forcing with overlapping extenders. Later in the week, on Wednesday, Sinapova presented Prikry iterations and its application on stationary reflection at $\aleph_{\omega+1}$. Then, on Thursday, Unger presented Gitik's diagonal overlapping extenders forcing and also the method of iterated ultrapowers. These new techniques give promising strategies for further analysis of singular combinatorics and many open problems.

Another topic was forcing axioms. On Monday and Wednesday, Schindler talked about his striking theorem with Aspero on MM^{++} and axiom $(*)$ and discussed further applications. On Tuesday, Todorcevic talked about OCA.

On Tuesday Neeman talked about the current state of the art of obtaining the tree property at long intervals of regular cardinals.

On the last day, the first talk was by Velickovic, who discussed guessing models. Guessing models are relevant both for applications of forcing axioms and the combinatorial nature of principles like the tree property and its strengthenings. The last morning talk, delivered by Zapletal, was about geometric set theory.

The talks were followed by group activities in the afternoon. During the afternoon of the first day, we had a problem session. We compiled a list of problems and people divided into groups accordingly.

Group discussions.
Below is a brief description of the work of each group.

MM(ω_1)

Participants: Natasha Dobrinen, John Krueger, Miguel Angel Mota, Menachem Magidor, Pedro Marun, Jindra Zapletal.

The question was whether the consistency of ZFC implies the consistency of Martin's Maximum restricted to partial orders of size ω_1 (MM(ω_1) for short). The group solved this in the affirmative.

First, Zapletal noted that the consistency of MM(ω_1) follows from the existence of an inaccessible cardinal. Later, the group realized that Shelah's general argument proving that the Semiproper Forcing Axiom implies Martin's Maximum can be adapted in such a way that we restrict ourselves to posets included in ω_1, and we only consider countable elementary substructures of $H(\omega_2)$. Therefore, it is possible to define- within ZFC+CH- a forcing iteration of length ω_2 using countable elementary substructures of $H(\omega_2)$ as side conditions, preserving cardinals and forcing MM(ω_1). It is worth noting that, in the presence of CH, this construction has the $\aleph_2$ chain condition since we require that our side conditions satisfy certain symmetry requirements which make them look like the entries of a certain infinite matrix. After a positive answer to the original question, the group discussed whether MM(ω_1) is in fact a consequence of PFA(ω_1).

Mota recently notified the workshop organizers that the group is about to submit a paper on the above work.

Welch games

Participants: Tom Benhamou, Sean Cox, James Cummings, Matt Foreman, Menachem Magidor.

Welch games have, as a background assumption, that κ is weakly compact, and have the following features:

(1) (Keisler-Tarski) κ is weakly compact iff Player II has a winning strategy for the game of length ω
(2) If κ is measurable then Player II has an easy winning strategy
(3) (Foreman-Magidor-Zeman) if Player II has a winning strategy for the game of length $\omega + 1$, then there is a precipitous ideal on κ.

The group looked into candidates for games that would play an analogous role, and have analogous properties, for larger cardinals (e.g. strong, supercompact). They believe they have good candidate games for the analogues for:

(1) λ-supercompactness of κ (with background assumption that κ is Π^1_1-λ subcompact).
(2) λ-strongness (with background assumption that κ is measurable)

In the latter case at least, Player II having a winning strategy for the game of length $\omega + 1$ should yield a precipitous "ideal extender" (in the sense of Claverie) with (generic) strength λ (so that if $j : V \to N$ is the generic elementary embedding by the ideal extender, then $V^V_\lambda \subset N$.)

Determinacy of these games (with large cardinals), even of the Welch games, is still open. The group plans to continue this collaboration in the future.

Undoing precipitousness

Participants: Paul Larson, Itay Neeman, and Martin Zeman participated all week; Natasha Dobrinen and Jindra Zapletal participated on the first day; Ralf Schindler participated the last two days.

The question was whether it is possible to force over a model where the non-stationary ideal on ω_1 is precipitous, to destroy the precipitousness of the ideal without adding subsets of ω_1.

A precipitous ideal is in some sense a pre-measure. A positive answer to the question would contrast with the situation for an actual measure. A negative answer would have meant that precipitousness is closer to measurability.

During the first day we noticed that, in contexts where covering holds, a positive answer could be very difficult, since the restriction of not adding subsets to ω_1 prevents us from adding subsets to anything that can be covered by a set of size ω_1. Already in the presence of measures, covering requires indiscernibles, and our first attempts involved trying to add new sequences of indiscernibles.

These attempts were not successful. But a later appeal to stronger large cardinals, specifically Woodin cardinals, led to an attempt using Woodin's extender algebra. The extender algebra is very flexible in that every object that exists externally to a model M can be made generic for the algebra in an iterated ultrapower of M. This allowed us to start with a model where the non-stationary ideal on ω_1 is not precipitous, force to make it precipitous (and in fact pre-saturated) over an intermediary model but not over the full universe, and then force with the extender algebra of an iterate of the intermediary model to absorb the failure of precipitousness.

This solved the question in the positive.

The solution still leaves some interesting questions open. Part of the motivation for the problem was to drive the development of techniques for adding small subsets to large cardinals (specifically in this case adding antichains in the non-stationary ideal on ω_1) without adding subsets of small cardinals. The solution to the problem did not go along these lines, as it used the extender algebra, which is combinatorially opaque. The original challenge remains to intentionally design a forcing that more transparently destroys precipitousness of the nonstationary ideal without adding subsets of ω_1. With the solution of the original problem we know that this is not asking for something impossible.

Can we obtain the tree property at $\aleph_{\omega_1+1}$ and at $\aleph_2$ simultaneously?

Participants: William Adkisson, Alejandro Poveda, Dima Sinapova, Spencer Unger.

This question is key to extending the tree property everywhere project past successors of singulars of uncountable cofinality. More precisely, the usual methods to get the tree property at $\aleph_{\omega_1+1}$ require a special choice of the cardinal that will become $\aleph_2$. Namely, in an inner model this cardinal, call it λ, must have been itself a successor of a singular. On the other hand, the only known way to get the tree property at $\aleph_2$ is to start with an inaccessible λ and turn it into $\aleph_2$.

So the two strategies are in conflict, since: if $V \subset W$, λ a successor of a singular in V, then λ cannot be inaccessible in W. The group formulated and discussed several test questions highlighting the difficulty:

(1) If $V \subset W$, λ a successor of a singular in V, $\lambda = \aleph_2^W$, then do we have weak square at λ in W?
(2) What other ways can there be to obtain the tree property at the successor of a singular?

Various forcings relevant to the above questions were discussed.

Working in the same group, Poveda asked whether it is consistent that the successor of a singular cardinal of uncountable cofinality can be fully supercompact in HOD. It was pointed out that this follows from the arguments in papers of either Gitik and Merimovich or Ben-Neria and Unger. The techniques of both papers do not extend to having two successors of singular cardinals being fully supercompact in HOD.

It also remains open to construct a version of "the AIM forcing" for uncountable cofinality. The main challenge is due to the fact that the known homogeneous posets for uncountable cofinalities are quotients of Magidor forcings. However, a promising approach is via nonstationary support iteration of supercompact Prikry/Magidor forcing.

Conclusion.
The workshop was a success. Participants learned about recent new developments in the combinatorics of infinite cardinals and various relevant forcing technique. A lot of open problems were discussed and strategies were outlined for problems that were previously intractable. A couple of the groups solved or came close to solving their problems. The workshop provided opportunities for young researchers to interact with experts from both of the main areas. We expect new collaborations to emerge.