Algorithms for lattices and algebraic automorphic forms

organized by
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Workshop Summary

The focus of the workshop was to establish the foundation for general methods of computing spaces of algebraic modular forms for a large class of classical reductive algebraic groups.

In the late 1990s, Gross [Gross] made a careful study of automorphic forms on reductive algebraic groups G over $\mathbb{Q}$ with the property that every arithmetic subgroup of $G(\mathbb{Q})$ is finite, dubbing these *algebraic modular forms*. As the name suggests, their study is completely algebraic in nature, owing to the fact that the underlying Shimura variety is zero-dimensional. The finiteness condition on the arithmetic subgroups of G seems more restrictive that it is: many split groups have inner forms with this remarkable property. The prototypical example is the group G of units in the $\mathbb{Q}$-algebra B of Hamilton's quaternions. In this case, G is an inner form of $GL_2/\mathbb{Q}$. The principle of functoriality implies, loosely speaking, that automorphic forms on G also arise on $GL_2/\mathbb{Q}$. These ideas have already been extended to a few new cases. In all these cases, computational methods for lattices play a key role. It seems, however, that today these ideas can be pushed much further both in theory and in practice, and this was the goal of the workshop.

The workshop brought together experts in number theory, arithmetic geometry, algebraic groups, and lattices—in both their theoretical and computational aspects. As usual for an AIM workshop, there were usually two talks in the morning. Highlights included a talk by Lassina Dembélé on his computations with Clifton Cunningham on Hilbert-Siegel modular forms over $\mathbb{Q}(\sqrt{5})$ using algebraic modular forms on a symplectic group, tying together many of the threads of the workshop, and a tag-team talk by Joshua Lansky and David Pollack taking a new look on the work they did as graduate students on algebraic modular forms for G_2 and $\mathrm{PGSp}_4(\mathbb{Q})$. Other topics included algorithms for lattices, Kneser's theory of neighbors, buildings, automorphic representations, and connections to coding theory and K3 surfaces. In between the morning lectures, there was an hour long gap for discussion. On Monday afternoon, a detailed list of problems (with a range of difficulty) was made, lasting much of the afternoon. In the remainder of the afternoon, we broke up into introductory groups: one popular such group was "p-neighbors for dummies", where a calculation was performed in the simplest case in careful and slow detail. On the other afternoons, the lionshare of effort was spent in small group collaboration, getting the attendees in different areas to come together to share their expertise. In place of a second talk on Friday morning, there was a presentation by each of the four working groups, as follows.

(1) *Algorithm to enumerate all (spinor) classes in the genus*

The heart of the proposed methods for computing algebraic modular forms involves enumerating the genus of a positive definite quadratic form over a totally real field using Kneser's method of $\mathfrak{p}$-neighbors. This has been carefully studied over the years, but there remains one sticky issue on explicitly enumerating the spinor genera inside the genus. Because strong approximation holds only on the spin group associated to the quadratic form and not the (special) orthogonal group, all $\mathfrak{p}$-neighbors remain inside just one of the spinor genera. In general, there will be many such classes, and in order to continue in an explicit way, one needs an algorithm to compute a set of primes that cover all spinor genera. This was the task of the first group.

For example from [**?**, 101.13], it is well known that the set of spinor genera in the genus of a lattice L is in bijection with J_F/PJ_F^L where J_F denotes the group of ideles of the base field F, P is the set of totally positive elements in F^* and

$$J_F^L = \{j \in J_F \mid j_\mathfrak{p} \in \theta(O^+(L_\mathfrak{p})) \text{ for all prime ideals } \mathfrak{p}\} .$$

Here θ denotes the so called spinor norm of the special orthogonal group $O^+(L_\mathfrak{p})$. If $\mathfrak{p}$ is an odd prime and it does not divide the volume $\mathfrak{v}(L)$, then $\theta(O^+(L_\mathfrak{p}))$ is well known to contain the local unit group $\mathcal{O}_\mathfrak{p}^*$. Otherwise it contains at least the squares. Moreover, the groups $\theta(O^+(L_\mathfrak{p}))$ can be computed explicitely.

But one computational problem remains. The group J_F/PJ_F^L is a quotient of two infinite groups that cannot be"constructed" in a computer algebra system easily. Hence the group looked into Magma's implementation for computing genus representatives for lattices over the rationals. It turns out that the implementation follows the beautiful description of [**?**, Section 11.3] which gives a description of J_F/PJ_F^L as a quotient of two *finite* groups. Unfortunately, this description does not carry over to arbitrary number fields.

However, inspired by the idea of Cassels, the group came up with the following description of J_F/PJ_F^L. Let $C_\mathfrak{m}$ denote the ray class group with respect to the ray $\mathfrak{m} := \infty \cdot \prod_{\mathfrak{p}|2\mathfrak{v}(L)} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(4)+1}$. Depending on the local groups $\theta(O^+(L_\mathfrak{p}))$ we can now explicitely write down generators for a subgroup S of $C_\mathfrak{m}$ such that $C_\mathfrak{m}/S$ is in bijection with the number of spinor genera in the genus. Hence it suffices to apply Kneser's neighbor method at a set of prime ideals that generates $C_\mathfrak{m}/S$.

We plan to implement these ideas in Magma in the following weeks; this is already nice tangible progress achieved at the workshop!

(2) *Orthogonal group calculations*

A second group focused on investigating some explicit calculations for orthogonal groups. Daniel Kim Murphy, a Ph.D. student of Akshay Venkatesh at Stanford, and John Voight (one of the organizers), both had written code before the workshop; Murphy's code was optimized to work over $\mathbb{Q}$ with trivial weight whereas Voight's code works over a general totally real field F and arbitrary weight. We shared ideas for the most efficient way to enumerate the $\mathfrak{p}$-neighbors: this boils down in part to a question on how to enumerate the isotropic spaces of a quadratic space over a finite field, but we noted that this itself does not completely determine the neighbors for the higher Hecke operators. Happily, the two different implementations (one in Sage the other in Magma) produced the same answers in all but one example, and the bug in the latter case was identified.

The group played with some explicit examples in low dimension, to identify the associated Galois representations in a classical way. We looked first at O(3) and a quadratic form of discriminant 11 over $\mathbb{Z}$: the corresponding space was identified with the classical space of forms of level 11 and weight 2: an Eisenstein series f and the modular form g associated to the elliptic curve of conductor 11. We then looked at several examples over O(4). In the first case, we looked at a quadratic form of conductor 11^2, and tracing the isogeny $\mathrm{O}(4) \leftrightarrow \mathrm{O}(3) \times \mathrm{O}(3)$, we identified the 3 forms as the possible Rankin-Selberg convolutions of the two forms f, g. We then looked at a quadratic form of conductor 29 and for the unique cusp form found the associated L-function as the Asai L-function of the unique Hilbert cusp form of level 1 over $\mathbb{Q}(\sqrt{29})$; this again has an explanation in terms of the isogeny of groups.

The next steps for this group would be to look at higher rank groups and identify the forms when they come from base change; it would be exciting to find some genuinely new forms! We agreed that it is time to be systematic about these calculations, and so to continue, we need a table of quadratic forms of small discriminants in a number of variables over totally real fields.

(3) *Unitary group calculations*

The third group concerned itself with computing spaces of algebraic modular forms for unitary groups. The members of this group discussed the general approach to such computations and reconciled two different versions of relevant code, one written by Sharlau and his students and one developed by Greenberg and Voight. Thanks to progress made at the workshop, we are now able to compute $\mathfrak{P}$-neighbors for arbitrary rank Hermitian lattices relative to an arbitrary CM extension and an arbitrary prime ideal $\mathfrak{P}$. Key to the implementation is Magma's facility for computing with number fields, their rings of integers, and finitely generated modules over these rings. We computed several modest-rank examples relative to cyclotomic extensions $\mathbb{Q}(\zeta_p)/\mathbb{Q}(\zeta_p)^+$ for $p = 3, 5, 7$. We analyzed the push-down to $\mathbb{Z}$ of the lattices we encountered and identified some very interesting p-modular lattices among these.

(4) *Neighbors and buildings*

A final fourth group worked to explicitly understand the relationship between neighbors and the building associated to the classical reductive group. They worked with Sp_4 over a local field and, by looking at the building, related the different Hecke operators to an action on the building in both the hyperspecial and nonhyperspecial cases. This work is technical and a bit consuming, but it promises some concrete implications for the implementation of Hecke operators on spaces of algebraic modular forms. In particular, further work needs to be done to see exactly what happens in more degenerate cases (farther away from hyperspecial) to see which neighbors correspond to which Hecke operators, analogous to the classical distinction between the Atkin-Lehner operator U_p and the Hecke operator T_p, each of which plays an important role in the theory.

Substantial work remains, especially in concrete implementation projects. We believe that the workshop will be only the beginning of many worthwhile collaborations and future lines of research on this topic.

Bibliography

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