Algorithmic randomness

organized by

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Workshop Summary

Aims and scope

This workshop was devoted to algorithmic randomness and its applications. Combining computability and probability, algorithmic randomness allows for a pointwise approach to many concepts that are classically measure-based, such as entropy and Hausdorff dimension. Recent results in the area indicate the applicability of algorithmic randomness to a wide variety of questions, from the fractal geometry of sets in Euclidean spaces to effective diophantine approximation. The workshop focused on these and other promising applications.

List of speakers

There were a small number of talks aimed at focusing discussion and providing background for the working groups, including three tutorials covering the main areas of the workshop, by Andrew Marks, Jan Reimann, and Jack Lutz / Don Stull / Neil Lutz.

- Jack Lutz, *Point-to-Set Principles*
- Neil Lutz, *Point-to-Set Principles Part III: Applications to Intersections, Products, and Hyperspaces*
- Andrew Marks, *Simultaneous Randomness*
- Elvira Mayordomo, *Effective fractal dimension in the hyperspace and the space of probability distributions*
- Jan Reimann, *An invitation to Fourier dimension*
- Alexander Shen, *Normality and finite state information theory*
- Frank Stephan, *Initial Segment Complexity for Measures*
- Don Stull, *Lines, Projections, and the Point-to-Set Principle*
- Marius Zimand, *Decouplers*

Working groups and results

The main focus of the workshop was discussion in the following small, focused groups.

Projection and Dimension.

This group focused on issues raised in Don Stull's tutorial talk. Point-to-set principles have allowed results in areas like fractal geometry to be proved using notions of partial algorithmic randomness. One of the strengths of this method is that it has in some cases allowed theorems to be extended from classes of well-behaved sets (from the point of view of descriptive set theory) to *all* sets. There are other cases in which such extensions cannot be obtained, however—and provably so—raising interesting set-theoretic issues.

The group discussed in particular Marstrand's First Projection Theorem, exploring connections with some recent work in descriptive set theory describing Hausdorff dimension in terms of games, and questions such as the minimum complexity of counterexamples to having this theorem hold for all sets. It also discussed Marstrand's Second Projection Theorem, whose proof involves Fourier analysis, and which does not currently have a proof using point-to-set-principles, raising the more general issue of how to combine point-to-set principles and Fourier analysis.

K.

K versus C C The group started work on the question of whether there is a constant bounding $C(C(x)|x, K(x))$ for all x. The first approach focused on a related question: is there a partial computable function ψ such that

$$(\forall x)\ [C(x) = \psi(x, K(x)) + O(1)]?$$

This should be easier to refute. We showed that the second question is equivalent to there being a choice of universal machines (plain and prefix-free) such that:

$$(\forall x)(\forall s)\ [K(x) = K_s(x) \to C(x) = C_s(x) + O(1)].$$

In words, once K is right, C must be right up to a fixed constant.

One can see the first equation as introducing a reducibility among functions: $f \leq_{pr} g$ if

$$(\exists \text{ partial computable } \psi)(\forall x)\ [f(x) = \psi(x, g(x)) + O(1)].$$

So we're asking whether $C \leq_{pr} K$. We showed that $K \not\leq_{pr} C$.

One can show that there is no largest function w.r.t. $\leq_{pr}$ among the functions that are computably approximable from above.

Simultaneous Randomness.

This group focused on an open problem brought to the workshop by Andrew Marks concerning whether Martin measure is strongly ergodic for Turing equivalence, via an approach that uses known theorems from ergodic theory about Borel probability measures, and the notion of reals being simultaneously random for a measure. Marks formulated an open question that could act as a test case for this approach. The group was able to come up with a negative answer to this question, but in a way that revealed an important obstacle that will likely need to be focused on in further work on the larger question.

The group made significant progress identifying what sort of measures one should be looking at, in particular noting that these need to be random measures. Ted Slaman made the intriguing observation that random measures seemed to come up in the discussions of several of the groups. The group identified a natural next step in this line of research, and a new test question involving 2-randomness that should help guide this work. At the group's final report, Marks said that he felt he understands much better where we should be looking for solutions to the overall problem following the work of this group, and is optimistic that a solution might be forthcoming using ideas arising during its discussions.

Fourier Dimension.

This working group discussed various aspects of an effective approach to Fourier dimension. The topics included the following:

Random closed sets and random measures: Measures of positive Fourier dimension often seem to come from randomized constructions. A good example is the Salem-Bluhm existence proof of a Salem set, which is based on distributing a mass along a randomly constructed closed set. Algorithmically random closed sets and measures have been studied extensively by Cenzer, Culver, Porter, and others. The group discussed whether, under a suitable framework, algorithmically random measures (or closed sets) can yield measures of positive Fourier dimension (and possibly Salem sets). It is also interesting to see how Jarnik's fractal fits into this context. This fractal has a deterministic definition not based on harmonic analysis but diophantine approximation properties, and is known to be a Salem set.

UD-randomness: Avigad has introduced a notion of *uniform-distribution (UD) randomness*. Avigad observed that every Schnorr random is UD-random, while there are reals that are not even Kurtz random but are UD-random. We studied Avigad's proofs and observed that Avigad's counterexample constructed a Π^0_1 class of Fourier dimension 1 but Lebesgue measure 0. We conjecture that in fact any real ML-random for a (computable) measure of positive Fourier dimension is UD-random.

Complexity of Measures.
We discussed *Martin-Löf absolute continuity* (ML a.c.), a notion of "randomness" for measures introduced by Nies and Stephan. For a measure μ and any $n \in \mathbb{N}$, let $H(\mu \upharpoonright n) = -\sum_{|x|=n} \mu[x] \log_2(\mu[x])$. With this definition in hand, we discussed:

Theorem (Bhojraj, unpublished)**.** *For any computable μ, consider the following properties:*
[topsep=4pt, itemsep=0pt]$(\exists c > 0)(\exists^\infty n)\ H(\mu \upharpoonright n) > n - c.$ *μ is ML a.c.* $H(\mu) := lim_n \dfrac{H(\mu \upharpoonright n)}{n} = 1.$
Then (1) ⇒ (2) ⇒ (3)*, and both implications are strict.*

We then discussed the following related topics:

1. Suppose one has a measure on the space of structures (in some language, with fixed countable underlying set) that is ML a.c. and invariant with respect to the logic action of S_∞ on this space. Then it must be the uniform measure. (Equivalently, if one has a property of structures such that the uniform measure on the orbit of a given structure has Lebesgue measure 0, then any computable S_∞-invariant measure on that orbit is non-ML a.c.)

2. We considered shift-invariance as it relates to entropy, and thought about whether we might generalize Theorem 3.3 of Nies and Stephan (https://arxiv.org/abs/1902.07871) to higher arity structures, or to consider extensions to $\mu-$ML a.c. with respect to a measure μ (perhaps when μ is maximal entropy in some sense).

3. We looked at Quinn Culver's notion of a uniform measure on measures, Mauldin and Monticino's variant of this work, and the connection between them that Chris Porter described.

Future plans

We anticipate an on-site follow-up AIM meeting on algorithmic randomness in 2022. The *Projection and dimension* group met after the meeting to discuss the answer to a question that had come up during the meeting. Some of its members have continued meeting since then and have plans to broaden the discussion to involve other members of the group, a mailing list having been created for this purpose. The *Fourier dimension* group continues to meet in Sococo and also established a Slack group for online discussion. The workshop also included a problem session resulting in a list of over twenty problems on a range of issues related to algorithmic randomness, which we expect workshop participants and other researchers will continue to work on.