Amenability of discrete groups

organized by
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Workshop Summary

Introduction

The workshop "Amenability of discrete groups," organized by K. Juschenko, T. Nagnibeda and V. Nekrashevych, was devoted to a number of active topics of research on amenability, such as structural properties of amenable groups, strong and weak forms of amenability, amenability of group actions, new examples of amenable and nonanemanble groups.

On the first day of the workshop, introductory talks were given in the morning and a problem session was held in the afternoon. About 50 questions were formulated and served as an inspiration for the following days activities. In the subsequent days we had two talks on the latest developments in the mornings and work in small groups in the afternoon. Every day about four problems were chosen from the problem list, to work on in groups. Some of the groups extended for a second and third day. The last afternoon was devoted to the reports from various working groups. We summarize below the results presented at this report session by the main working groups.

Finite presentations of groups

Let $\mathfrak{G}$ be a groupoid of germs of an action of a finitely generated group G on a Cantor set $\mathcal{X}$. Denote by $\mathsf{A}(\mathfrak{G})$ the group generated by even permutations of disjoint clopen subsets of $\mathcal{X}$ such that all their germs belong to $\mathfrak{G}$.

We know that $\mathsf{A}(\mathfrak{G})$ is finitely generated if and only if the action is expansive and has no small orbits (orbits of length 2 or less).

We proved at the conference the following theorem.

Theorem. *Let N be a normal subgroup of $\mathsf{A}(\mathfrak{G})$, and let F be the set of points of $\mathcal{X}$ fixed by all elements of N. Then N contains $\mathsf{A}(\mathfrak{G}|_{F^c})$ and is contained in the pointwise fixator of F.*

In particular, this theorem allows to describe the profinite completion of $\mathsf{A}(\mathfrak{G})$ in the case when $\mathfrak{G}$ is the groupoid of germs of the action of a $\mathbb{Z}$-shift. Namely, it is isomorphic to the direct product of the groups $H_n \rtimes \mathsf{A}_n$ over all cycles of the shift, where n is the length of the cycle, $H_n < \hat{\mathbb{Z}}^n$ is the subgroup of elements with zero sum of the coordinates, and $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$.

We have also proved the following.

Theorem. *If $\mathfrak{G}$ is the groupoid of germs of the action of a $\mathbb{Z}$-shift. If the shift is not of finite type, then the group $\mathsf{A}(\mathfrak{G})$ is not finitely presented. If $\mathfrak{G}$ is the groupoid of germs generated by a one-sided shift, then $\mathsf{A}(\mathfrak{G})$ is finitely presented if and only if the shift is of finite type.*

The restrictions on existence of finite presentations for shifts of infinite type is generalizable to other types of groupoids.

Strongly amenable groups

The task was to find more examples of groups that are not strongly amenable, in particular in relation to the question whether strongly amenable groups are necessarily virtually nilpotent.

New examples of non strongly amenable groups were found at the conference. In particular, we have found a locally finite not strongly amenable group. It is the lamplighter group on the infinite (locally finite) symmetric group S_∞. One can show that it has a proximal action on the quotient of the Cartesian power $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ by the involution $x \mapsto x+1$ acting on all coordinates.

We also have found an example of a non strongly amenable group of intermediate growth. It is a simple group of the form $\mathsf{A}(\mathfrak{G})$ of intermediate growth. The construction of a proximal action is similar to the construction for the Thompson group as in the recent work of Hartman, Juschenko, Tamuz and Vahidi Ferdowsi that was reported on during the conference.

Cantor-Bendixon rank of the Thompson group F

The question is to understand the Cantor-Bendixon rank of the Chabauty space of all subgroups of the Thompson group F. The Cantor-Bendixon rank of a topological space is by definition the smallest ordinal of steps needed to reach the perfect kernel of this space by removing at each step the set of isolated points.

We have shown at the conference that the rank is at least ω_0, and reached some understanding of subgroups of low rank and which groups should belong to the perfect kernel (i.e., which groups have uncountably many groups in every neighborhood). In particular, some evidence was obtained that allows to conjecture that all infinitely generated subgroups belong to the perfect kernel, while finitely generated ones may either have finite CB rank or belong to the perfect kernel.

Liouville actions

The goal was to understand if there exists a transitive action $G \curvearrowright X$ with amenable stabilizers of a non-amenable group G which is Liouville with respect to some probability μ on G.

We realized that such actions exist even with abelian stabilizers. The construction is as follows. Consider a group G that can be factorized into a product $H \cdot K$ of its subgroups so that H acts transitively on G/K. If H is amenable, then one can find a Liouville measure for the action $G \curvearrowright G/K$ using the Vershik-Kaimanovich trick. The stabilizers of the action are conjugate to K, so if K is amenable, they are amenable. As an example, consider $SL_2(\overline{\mathbb{Q}})$

factorized with $K = SO_2(\overline{\mathbb{Q}})$ and H equal to the group of upper triangular matrices. Then H is solvable, and K is abelian.

Natural SQ-closed classes of amenable groups

A class of groups is called SQ-closed if it is closed under taking subgroups and quotients. For example, the class of virtually abelian groups or the class of groups of sub-exponential growth are SQ-closed. If $\mathfrak{B}$ is an SQ-closed class of amenable groups, then it is natural to consider the class of $\mathfrak{B}$-*elementary amenable groups* consisting of all groups that can be obtained from $\mathfrak{B}$ using the operations preserving amenability (extensions and direct limits).

It seems that the following classes are interesting from the point of view of the theory of amenability.

We say that a group G is SL if it is Liouville for every finitely supported probability on G. We proved that SL is an SQ-closed class containing all groups of subexponential growth and all groups generated by bounded automata. It seems that currently there are no known examples of amenable groups that are no SL-elementary amenable.

For example, the following problem seems to be open.

Does there exist a finitely generated weakly branch amenable group not belonging to SL? It will be not SL-elementary amenable in that case. There are examples of non-Liouville contracting self-similar groups. They do not contain free subgroups, but it is not known if they are amenable.

Another natural way to define SQ-classes is considering asymptotics of the return probabilities for the simple random walk. One defines the class R_γ of groups for which the return probability in $2n$ steps is bounded from above by a function $\gamma(n)$. Then one can consider the elementary closure of the class R_γ. We do not know how this class depends on γ, for example, whether we get different classes for different functions.