The uniform boundedness conjecture in arithmetic dynamics

organized by
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Workshop Summary

The workshop was a great success, generating a large amount of mathematical activity, including new theorems, new problems, and new approaches to solving old problems. The field of arithmetic dynamics is fairly new and the people working in the area form a fairly small group. It was thus exciting to gather a large proportion of the experts and to mix them with a lot of young postdocs and graduate students who are entering the field, together with a number of people who work in related areas. There were many comments, especially among the younger people, along the lines of "I've read many of your papers, I'm so glad to get a chance to meet you."

It was also exciting to find out that the recently announced draft for the 2010 Mathematical Sciences Classification scheme includes a new section, 37Pxx, devoted to arithmetic and non-archimedean dynamics. AIM's workshop is thus at the forefront of a new and burgeoning area of mathematical research.

The morning talks at the workshop were very useful, both for bringing people up to speed and for introducing people to ideas, techniques and results for use in studying arithmetic dynamics in general and uniform boundedness in particular.

However, the centerpiece of the week was the huge amount of activity and excitement generated by the afternoon working sessions. In order to convey what was accomplished, here are brief descriptions of some of the group sessions. These descriptions use the following notation, which was more-or-less constant throughout the workshop:

ϕ	a rational map $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$.
ϕ_c	the polynomial $\phi_c(z) = z^2 + c$.
$\mathrm{Per}(\phi)$	the set of periodic points of ϕ.
$\mathrm{PrePer}(\phi)$	the set of preperiodic points of ϕ.
$\Phi_N(c,z)$	the "dynatomic" polynomial $\prod_{n\mid N}\bigl(\phi_c^n(z) - z\bigr)^{\mu(N/n)}$ whose roots are the points of formal period N for $\phi_c(z)$.
$Y_1^{\mathrm{dyn}}(N)$	the affine curve $\Phi_N(c,z) = 0$ in $\mathbb{A}^2$. Its points classify pairs (c,z) such that z is a point of formal period N for ϕ_c.
$X_1^{\mathrm{dyn}}(N)$	the smooth projective closure of $Y_1^{\mathrm{dyn}}(N)$.
$Y_0^{\mathrm{dyn}}(N)$	the quotient of $Y_1^{\mathrm{dyn}}(N)$ by the automorphism $(c,z) \to (c, z^2 + c)$.
$X_0^{\mathrm{dyn}}(N)$	the analogous quotient of $X_1^{\mathrm{dyn}}(N)$

The *cusps* of $X_1^{\mathrm{dyn}}(N)$ are the points in the complement of $Y_1^{\mathrm{dyn}}(N)$, and similarly for $X_0^{\mathrm{dyn}}(N)$. The cusps can be described in terms of symbolic dynamics.

0.1 Uniform boundedness over function fields

Let $K = k(C)$ be the function field of a smooth projective curve C/k of genus g. The genera of $X_0(N)$ and $X_1(N)$ are known and grow with N (theorems of Bousch and Morton), so uniform boundedness is trivial if one allows the bound to depend on the genus of C. However, the function field analog of the strong form of uniform boundedness asks for a bound depending only on the gonality of C. (The gonality of a curve is the smallest degree of a map to $\mathbb{P}^1$.) Bjorn Poonen came up with a proof, based on the Castelnuovo-Severi inequality, so that is a new theorem. However, the bound has weak dependence on N. A working group came up with a promising approach for a stronger bound based on the behavior of the cusps when reduced modulo p.

0.2 Measures of bad reduction

Three of the groups on the first day, each of which was working on a distinct problem, ended up at more-or-less the same place. They all needed to formulate an appropriate notion for, and measure of, "bad reduction" that could be directly related to the possible degenerations of the dynamics of ϕ_c. There were a number of ideas proposed, including the primes for which the curve $X_1^{\mathrm{dyn}}(N)$ has bad reduction, the primes for which the fibers over the cusps or over other points come together, the minimal discriminant of ϕ_c^N, etc. None of these seems entirely satisfactory, but before the workshop it hadn't been clear just how important it will be to create a framework that can be used to measure bad reduction and relate that measurement in a precise way to the geometry of the dynamical system.

0.3 Precritical dynamical modular curves

Rather than studying periodic and preperiodic points, which are solutions to $\phi_c^i(z) = \phi_c^j(z)$, one can instead study the backwards orbit of the finite critical point $z = 0$. The full critical orbit plays a key role in classical complex dynamics, so it is natural to study the arithmetic properties of this orbit. One defines the precritical dynamical modular curve $C_1^{\mathrm{dyn}}(N)$ to be a nonsingular model for the affine curve $\phi_c^N(z) = 0$. The working group made a large amount of progress over the course of three days and was still going strong as the workshop ended. Among their results: (1) genus, gonality and ramification formulas for $C_1^{\mathrm{dyn}}(N)$. (2) uniform boundedness for rational precritical points (which follows from the fact that the curves lie in a tower). (3) explicit computations on the genus 5 curve $C_1^{\mathrm{dyn}}(4)$, with the hope that soon there will be an exact uniform boundedness theorem. The group also considered analogous questions for $z^3 + c$ and, more generally, $z^d + c$.

0.4 The transition from preperiodic to periodic

This group studied the extent to which there can be strictly preperiodic points, in particular over p-adic fields. They proved a number of interesting results and formulated some conjectures that will certainly generate more work. Among the more striking results that they proved is the existence of polynomials that have arbitrarily long preperiodic tails of points all defined over $\mathbb{Z}_p$.

0.5 The cuspidal subgroup and dynamical units

The divisors of degree 0 supported on the cusps of $X_1^{\mathrm{dyn}}(N)$ generate a subgroup $V_1^{\mathrm{dyn}}(N)$ of the Jacobian variety $J_1^{\mathrm{dyn}}(N) = \mathrm{Jac}\big(X_1^{\mathrm{dyn}}(N)\big)$, and similarly for $X_0^{\mathrm{dyn}}(N)$. In the classical (elliptic) setting, the cuspidal subgroup is finite, which plays a key role in Mazur's proof of uniform boundedness of torsion on elliptic curves. It is thus of interest to study the dynamical cuspidal subgroups. The number of cusps is known, so the question becomes that of determining

$$K[Y_1^{\mathrm{dyn}}(N)]^*/K^* = \mathrm{Kernel}\Big(\mathrm{Div}^0_{\mathrm{cusp}}(X_1^{\mathrm{dyn}}(N)) \longrightarrow J_1^{\mathrm{dyn}}(N)\Big).$$

There is an earlier construction of so-called dynamical units, which are defined by $u_{ij} = \big(\phi_c^i(z) - \phi_c^j(c)\big)/\big(\phi_c(z) - z\big)$. The group determined that for $N = 5$, the dynamical units are maximally independent, which puts an upper bound on the rank of $V_1^{\mathrm{dyn}}(5)$. This also raised the interesting question of whether the dynamical units generate a subgroup of finite index in $K[Y_1^{\mathrm{dyn}}(N)]^*/K^*$, and if so, what is that index.

0.6 Elementary approaches

Bjorn Poonen came up with an elementary sounding problem whose solution would imply uniform boundedness. He also came up with an elementary combinatorial problem that might be used to improve Benedetto's $O(s \log s)$ bound for $\#\,\mathrm{PrePer}(\phi, K)$ when ϕ has bad reduction at s primes. These ideas were explored in two working groups, and although no results were proven, the ideas generated are well worth pursuing.

0.7 Dynamical Mordell-Lang conjecture

After Tucker's talk on dynamical Mordell-Lang conjecture (DML), a working group formed to discuss how one might prove further cases of the DML conjecture. In particular, a key tool in the recent work of Tucker et al. was Rivera-Letelier's deep p-adic uniformization theorem, so the group investigated how one might use this and similar results in new DML situations.

0.8 Berkovich space

Talks by Rivera-Letelier and Baker set the stage for a session on how one might use equidistribution results for Berkovich space to prove various sorts of uniform boundedness results. A key observation made at the conference was that in order to do this, one will need explicit formulas for the Lipschitz and Hölder constants that appear in the equidistribution inequalities. Work on doing this, which was started at the conference, is continuing.