Arithmetic statistics, discrete restriction, and Fourier analysis

organized by
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Workshop Summary

Workshop Report

Arithmetic statistics, discrete restriction, and Fourier analysis

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Topics discussed

This workshop was in a handful of areas related to analytic number theory and harmonic analysis. It was the organizers' belief that researchers in these fields spend too little time talking to each other, and that an AIM workshop was the ideal venue for number theorists and analysts to get together, sit in the same room (or "room" – because of the pandemic, the workshop was held online via Sococo), and share ideas. In our estimation the workshop was quite the success, as evidenced by the project groups making good progress on relevant problems, and consisting of researchers in different areas who in many cases had not colloaborated before the workshop.

Lectures. We had two lectures each morning, given (in order) by the following people: Robert Lemke Oliver, Kevin Hughes, Emanuel Carneiro, Zane Li, Yu-Ru Liu, David Lowry-Duda, Alexandra Florea, Kirsti Biggs, Eyvindur Palsson, and Eun Hye Lee. Speakers were chosen to address a variety of techniques and questions encompassing (more or less) all of the topics of the workshop, and were disproportionately junior.

We omit a more detailed description, as video is available from the AIM website.

Problem session. We held a problem session on Monday afternoon, moderated by Caroline Turnage-Butterbaugh, and transcribed by George Shakan. What follows are George's notes on several of the problems proposed, which participants did not end up working on this week.

Problem 1 (J. Wang)**.** *Wright's approach meets harmonic analysis.*

Classically enumerating abelian extensions over k (equivalently homomorphisms from the idele class group of k) is done by Wrights via studying the generating series of abelian extensions. Recently (for example see work of https://arxiv.org/abs/1508.02518Christopher Frei, Daniel Loughran, and Rachel Newton) it is reproduced via applying Poisson summation to such generating series and studying the summation of Fourier transforms. How much can we gain from moving to the Fourier side? For example, are there cases where a generating series that cannot be written as a finite sum of Euler product becomes a finite sum of Euler product on the Fourier side?

Problem 2 (Hughes/Thorne). *Prove equidistribution of* $n^2\alpha \mod 1$ *without Fourier techniques. Or perhaps understand better why Fourier techniques appear.*

Let $f : \mathbb{Z} \to \mathbb{C}$. Set

$$A_N(f)(x) = \frac{1}{N}\sum_{n=1}^{N} f(x-n^2).$$

Let

$$Mf(x) = \sup_{N\in\mathbb{Z}_{\geq 1}} |A_N(f)(x)|.$$

Bourgain proved, using Fourier analysis and the circle method, that M is bounded on ℓ^p for all $p > 1$. Kevin wanted to know another proof of this fact for any finite p.

Problem 3 (Anderson). *Let* Q *be a nonsingular, indefinite quadratic form. Provide sharp discrete restriction estimates.*

For a concrete choice, one may consider

$$Q(x_1, \ldots, x_{m+1}) = \sum_{j=1}^{m} x_j^2 = \sum_{j=m+1}^{n} x_j^2,$$

with $n \geq 4$ and $2 \leq m \leq n-2$. One can consult https://arxiv.org/abs/2004.02301this paper of Cook, Hughes, and Paulson for some progress, as well as works of Bourgain–Demeter and Henriot–Hughes, on this problem.

Problem 4 (Thorne). *Consider the Shintami zeta function (associated of lattice of binary cubic forms). There is no Euler product. Can one bound the size of its coefficients. Here the coefficients count the number of certain cubic rings.*

He warns that this seems quite hard.

Problem 5. *Study* $f(t) := \sum_{n=1}^{N} a_n n^{it}$*, in particular, for integer* $k \geq 1$*, let*

$$\int_T^{2T} |f(t)|^{2k} dt.$$

It is known this is

$$\ll (T + N^k)\left(\sum_{n=1}^{N} |a_n|^2 \tau_k(n)\right)^k.$$

Perhaps more precise estimates? What about Mongomery's conjecture which is for $1 \leq k \leq 2$.

This is a multiplicative analog of discrete restriction and the point is that perhaps there could be some new input from harmonic analysts here. This is not true for non-integer k. See https://terrytao.wordpress.com/2015/02/13/254a-notes-6-large-values-of-dirichlet-polynomials-zero-density-estimates-and-primes-in-short-intervals/this post of Tao and Bourgain's linked paper. See also page 232 of Iwaniec and Kowalski.

Problem 6 (Wooley). *Discrete restriction over function fields.*

Problem 7 (Carneiro)**.** *Compute*

$$C = \inf_{f \in \mathcal{A}} \frac{||f||_1}{|f(0)|},$$

where $\mathcal{A}$ *is the set of continuous, integrable* $f : \mathbb{R} \to \mathbb{R}$ *with* $\mathrm{supp}(\widehat{f}) \subset [-1, 1]$.

It is known that

$$.857.. < C < .9259....$$

Such f are so-called bandllimited functions.

Problem 8 (Carneiro)**.** *Fix* $K \subset \mathbb{R}^d$ *a symmetric convex body. Compute*

$$\inf_{f \in \mathcal{A}} \int F(x)dx,$$

where $\mathcal{A}$ *is the set of continuous, integrable* $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ *with* $F(0) \geq 1$ *and* $\mathrm{supp}(\widehat{F}) \subset K$*.*

Problem 9 (Carneiro)**.** *Let* $\lambda_1, \dots \lambda_N \in \mathbb{C}$ *such that* $|\lambda_m - \lambda_n| \geq \delta_n$*. Is it true that*

$$\sum_{n,m=1}^{N} \frac{a_n \overline{a_m}}{\lambda_n - \lambda_m} \leq \pi \sum_{n=1}^{N} \frac{|a_n|^2}{\delta_n}?$$

This is known if the common differences is a constant. Note this is a linear algebra problem, which asks to bound the spectrum of a certain matrix. The constant $4\pi/3$ replacing π is known by Pressman, improving earlier work of Montgomery and Vaughan.

Project groups

The heart of any AIM workshop is the project groups, and participants split into five groups to work on a variety of problems. Each of the groups was asked for a report, and these reports follow.

Discrete restriction for (n, n^3)**.** (Based on a longer report prepared by Bingyang Hu.) Bingyang Hu, Kevin Hughes, Jose Madrid, Frank Thorne, and Trevor Wooley looked at a question involving discrete restriction.

The model case is a recent discrete restriction estimate due to Hughes and Wooley [HW], in which, they adapted an elementary arithmetic counting method to improve the upper bound the $L^{10}(\mathbb{T}^2)$-norm of the exponential sum

$$E\mathbf{a}(\alpha, \beta) := \sum_{|n| \leq N} a(n) e\left(\alpha n^3 + \beta n\right).$$

After reading this paper and some of the related literature, the group decided to investigate the possibility of generalization to other cases. We have tried to improve the $L^{14}(\mathbb{T}^2)$ norm of the exponential sum

$$E\mathbf{a}'(\alpha, \beta, \gamma) := \sum_{|n| \leq N} a(n) e\left(\alpha n^5 + \beta n^3 + \gamma n\right)$$

by using a similar approach as Hughes and Wooley. In this case, we determined that we could not beat the the nested efficient congruencing bound (or the decoupling bound). Our next step, suggested by Wooley, is to investigate the sum

$$\sum_{|n|\le N} a(n)e\left(\alpha_k n^{2k-1} + \alpha_{k-1}n^{2k-3} + \cdots + \alpha_1 n\right)$$

for certain choice of p which beats the decoupling bound. We have reason to believe this is feasible and might have a preprint to share in the coming future.

Fourier optimization and equidistribution of zeros of polynomials. (Based on a longer report prepared by Micah Milinovich.) Emanuel Carneiro, Mithun Das, Alexandra Florea, Angel V. Kumchev, Amita Malik, Micah Milinovich, Caroline Turnage-Butterbaugh, and Jiuya Wang worked on a Fourier optimization problem proposed by Carneiro.

Following the elegant treatment of Soundararajan [Sound], we revisited the classical work of Erdős and Turán [ET] on the distribution of zeros of polynomials in the complex plane. Following the notation in [Sound], let $P(z) = \prod_j^n \left(z - \alpha_j\right) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a monic polynomial of degree n with roots $\alpha_j = \rho_j e^{i\theta_j}$, and define its *height* as

$$h(P) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{\left|P(e^{i\theta})\right|}{\sqrt{|a_0|}}\, d\theta,$$

where $\log^+(x) = \max(0, \log x)$. Given an arc I on the unit circle, let $N(I;P)$ denote the number of zeros α_j with $e^{i\theta_j}$ lying on this arc. If the θ_j are equidistributed, then $N(I;P)$ should be roughly equal to $|I|n/(2\pi)$, where $|I|$ denotes the length of I. We define the *discrepancy* as

$$\mathcal{D}(P) = \max_I \left| N(I;P) - \frac{|I|}{2\pi}\, n \right|.$$

Soundararajan proved that $\mathcal{D}(P) \le \frac{8}{\pi}\sqrt{h(P)\deg(P)}$ and speculated that "there is some scope to improve the constant $8/\pi$." It is also known [AM] that $\limsup_{\deg(P)\to\infty} \dfrac{\mathcal{D}(P)}{\sqrt{h(P)\deg(P)}} \ge \sqrt{2}$.

In brief, the group tried to improve these two results by improving the choice of g. This project already met with success during the week of the workshop and the group intends to continue its work.

Fourier transforms related to polynomials over finite fields. (Prepared by Theresa Anderson.) Theresa Anderson, Ayla Gafni, Kevin Hughes, Robert Lemke Oliver, George Shakan, Hong Wang, Jiuya Wang, and Ruixiang Zhang worked on a project aimed at bringing together arithmetic statistics and Fourier analysis via computing Fourier transforms of characteristic functions related to counting polynomials. Using Fourier analysis to improve or expand counting problems related to algebraic objects ("arithmetic statistics") was one of the goals of this workshop, and our group is pursuing a problem in this direction. Motivated by the Hilbert Irreducibility Theorem, we hope to make progress toward the following conjecture (or variants thereof):

Conjecture 1. *Let $P_n(H) = \{f(x) \in \mathbb{Z}[x] := V : f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0, |a_i| \le H\}$. Then the number of f with Galois group not isomorphic to S_n is bounded above by a constant times $|P_n(H)|/H$.*

In particular, we hope to make improvements to the best known bounds under the additional assumption that $\mathrm{Gal}(f)$ is not isomorphic to A_n, as this is the case that is least well understood. With an eye to applying the Selberg sieve, we arrive at estimating the following count

$$\sum_{f \in P_n(H)} \chi_q^{odd}(f)$$

where χ_q^{odd} is the characteristic function of polynomials with odd factorization type mod p for all p dividing q (q squarefree). The reason we are looking at this count stems from wanting to sieve out polynomials corresponding to odd permutation types which will correspond to A_n polynomials mod p. In applications, we may smooth out this sum via multiplication by a Schwartz function. Once we have this sum, the Fourier analysis arises due to applying Poisson summation, which converts this problem to the Fourier side, and leaves us to estimate $\widehat{\chi_q^{odd}}$ on the dual space V^*, and good estimates should come by computing what this Fourier transform is on orbits of the group action $\mathrm{GL}_1(\mathbb{F}_q) \times \mathrm{GL}_2(\mathbb{F}_q)$, due to an invariance of the orbits under the group action that is compatible with taking Fourier transforms. We care about estimating nonzero Fourier coefficients $\widehat{\chi_q^{odd}}$ in particular for large q ("the error term"). We have several ideas on how to proceed. We hope that our methods and ideas related to using these computations as well as the computations themselves will inspire similar techniques in other counting problems.

Spherical maximal functions in dimension four. (Prepared by Kevin Hughes.) Theresa Anderson, Julia Brandes, Ayla Gafni, Kevin Hughes, Angel Kumchev, and Eyvindur Palsson worked on a project concerning spherical maximal functions.

A classical operator in analysis is the spherical averaging operator and its associated maximal function which were introduced in celebrated work of Elias Stein. Our project concerns a discrete analogue of Stein's spherical maximal function in four dimensions. In dimensions five or more this problem was introduced by Akos Magyar and a full solution was given in an influential work of Akos Magyar, Elias Stein and Stephen Wainger. Alexandru Ionescu produced an example which showed that the natural analogue of Magyar's discrete spherical maximal function in four dimensions fails to be bounded on any $\ell^p(\mathbb{Z}^4)$ for $p < \infty$. However, Kevin Hughes showed that the underlying issue in that maximal function was a 2-adic obstruction. Moreover, by modifying the maximal function to remove that obstruction, he showed that a dyadic version of the modified discrete spherical maximal function in L^2 is bounded with only a 'logarithmic-loss'.

Hughes went on to conjecture that the modified discrete spherical maximal function is bounded on $L^p(\mathbb{Z}^4)$ for all $2 < p \leq \infty$ in analogy with Bourgain's result for Stein's spherical maximal function in $\mathbb{R}^2$. Analytical complications arise as this maximal operator fails to be bounded on the critical ℓ^2 space; this mirrors the challenges faced by Bourgain when studying the continuous counterpart since fundamental Fourier analytic techniques are diminished. Hughes's conjecture is supported, though not proven, by his recent work in discrete L^p-improving inequalities. By analogy with work of Mockenhaupt–Seeger–Sogge, Hughes suggests that we prove the dyadic modified discrete spherical maximal function in four dimensions is bounded on $\ell^4(\mathbb{Z}^4)$ with a bound independent of the dyadic range. The beauty of ℓ^4 is that one can view it as a square, squared and thus allows for Fourier analytic techniques, although with significant complications. We plan to explore this in our first meeting following this AIM workshop.

Möbius randomness and discrete restriction. (Prepared by Zane Li.) A large group – Theresa Anderson, Kirsti Biggs, Julia Brandes, Ayla Gafni, Kevin Hughes, Zane Li, Yu-Ru Liu, Amita Malik, Eyvindur Palsson, Lillian Pierce, Andrei Shubin, Alexander Walker, and Trevor Wooley – worked on a problem involving Möbius randomness and discrete restriction. Assume that a_n is uncorrelated with $\mu(n)$. Do we have

$$\int_0^1 |\sum_{n\le X} a_n\mu(n)e(\alpha n^2)|^4\,d\alpha \ll (\sum_{n:n \text{ squarefree}} |a_n|^2)^2 \tag{1}$$

or

$$\int_0^1\int_0^1 |\sum_{n\le X} a_n\mu(n)e(\alpha n^2+\beta n)|^6\,d\alpha\,d\beta \ll (\sum_{n:n \text{ squarefree}} |a_n|^2)^3? \tag{2}$$

The first day we considered instead the exponential sum $\sum_{n\le X} a_n\mu(n)e(\alpha n)$ and looking at applying the TT^* method in analyzing its L^4 norm. Another question we considered was whether in (1) and (2) could we just take $a_n = 1_A(n)$ for an arbitrary subset of $\{1, 2, \dots, X\}$. From dyadic decomposition, we may assume that $a_n\mu(n) = 1_A(n)$, however it seems that some care is needed to assume that $a_n = 1_A(n)$.

Later in the week, we decided to restrict to the case when $a_n = 1$ and use that 4 and 6 are even. In the case of $a_n = 1$ in (2), using translation invariance of the quadratic Vinogradov system and averaging over shifts of such solutions, we were led to consider

$$\sum_{\substack{1\le n_1,\dots,n_6\le X\\ n_1+n_2+n_3=n_4+n_5+n_6\\ n_1^2+n_2^2+n_3^2=n_4^2+n_5^2+n_6^2}} \frac{1}{H}\sum_{h\le H}\mu(n_1+h)\cdots\mu(n_6+h)$$

where we think of H as a power of X. The average over h now potentially looks like something where we could imply some conjectural estimates about Möbius autocorrelation.

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