Time-dependent Bernoulli-type free boundary problems

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Workshop Summary

The workshop aimed at studying geometry properties of Bernoulli Free boundary problems in dynamic setting, motivated by recent advances made on mean curvature flows and capillary flow. The problems discussed during the workshop fell into roughly two categories, namely (i) Understanding special class of solutions to the problem in the spirit of blow-up profiles, (ii) Understanding the generic nature of the singularities in the problem, and (iii) Understanding the free boundaries in the context of capillary drops.

Below we report on progress of the group projects initiated at AIM and future plans.

Classifying special solutions

A large group decided to work on the problem of classifying special solutions, which was somewhat open ended. The group naturally ended up dividing into a number of subgroups. We will start the report with a summary and then continue with detailed reports from the sub-groups.

The groups working on "classification" explored several related issues concerning self-similar solutions to the parabolic Bernoulli flow. The questions concerning self-shrinkers (backward self-similar solutions) include:

(1) Is the only self-shrinker with compact, convex support the ball?
(2) What is the meaningful notion of mean-convex here (following the analogy to mean curvature flow), and what are the mean-convex self-shrinkers?
(3) What are the radial self-shrinkers?
(4) Are there self-shrinkers in other geometric configurations, like with co-compact support, asymptotic to minimal cones, etc.?

Groups have made partial progress on the second question and almost completely answered the third. The rest are still open despite our efforts.

On the topic of translators (traveling waves), there are analogous questions:

(1) What are the translators with convex support or co-convex support in 2D? Are there "grim reaper" translators, in the sense that their support is trapped between two parallel lines?
(2) Are there nontrivial translators in higher dimensions? We have made some partial progress on the first question, eliminating "grim reaper" solutions under some geometric assumptions, but the larger question remains open. In particular, we now believe there are nontrivial traveling waves in 2D which look closer to the mean curvature "bowl soliton" than the grim reaper.

Some discussions on these questions are ongoing, though most of them seem to be quite hard. I expect that while there may be progress in the future, what will happen in the shorter term is several papers by some subsets of those who attended the workshop on adjacent topics, like regularity and blow-up analysis for the flow. The discussion so far has highlighted both the importance of this overall topic and the lack of nonlinear PDE estimates available for the flow, despite other technologies being available. In the mean curvature flow literature, there is a close connection between estimates on the flow and proofs of rigidity of self-similar solutions, but those estimates do not carry over cleanly to the free boundary context.

Translators.
Translators form an important class of solutions to the parabolic Bernoulli problem. They can be used as barriers for general flows and as models for 'type II' blow ups. At the workshop, our group mainly focused on the two-dimensional case. We were able to write down several examples, and we were able to rule out 'grim reaper'-type solutions. Other than that, the behavior of translators can be quite rich, and a classification, even in two dimensions, seems out of reach. The group studying translators overlaps significantly with the group studying self-shrinkers. At this stage, it seems that a classification of self-shrinkers is a more meaningful project. The group decided to concentrate more on self-shrinkers after the workshop.

Radial Solutions.
Inspired by developments on the mean curvature flow, we introduced the notion of Type-I singularity and show that these singular points are infinitesimally modeled on self-similar solutions. We also construct and classify all such self-similar solutions which are radially symmetric: their positivity sets are a unique ball, a unique annulus, or the exterior of a ball. A natural next step for our project is to analyze the local behavior of the solution to the parabolic Bernoulli problem near a Type-I singular point, given that the solution blows up to one of the self-similar solutions with compact support, such as studying the uniqueness of the tangent flow, analyzing the local asymptotics with respect to the self-shrinkers, etc.

Mean Convex Flows.
As mentioned above this group set out to develop a meaningful notion of "mean convex" flow in the Bernoulli setting. The group seems to have found a suitable notion that is preserved along the flow, extending some previous results in the literature. The group is currently meeting to work out issues of existence and regularity, with the hope of proving singular set estimates in space time for such flows analogous to those in mean convex mean curvature flow. However, the group has not made any progress on the goal of classifying mean convex self-shrinkers.

Hydrophilic limit of Capillary MCF

Since the workshop, the group working on the Hydrophilic limit of mean curvature flow (MCF) has met once. The group met to revisit some of the ideas developed during the workshop. The general framework of the problem is to consider MCF with vanishing Young's angle. Starting from compatible initial data, we aim to show that solutions of MCF converge to a solution of the parabolic Bernoulli equation. Under the assumption that the MCF data is graphical and mean convex, we believe we have a coherent proof strategy. This will rely on showing that solutions of the flow are non-degenerate at the free boundary. To obtain this, we can pair that the mean convex initial data will lead to outward minimizing states at all times along with a nondegeneracy estimate due to B. Orcan. Ultimately, we hope to show starting from non-graphical data it becomes graphical on some quantitative time-scale at which we can deduce the hydrophilic limit of MCF is still the parabolic Bernoulli equation.

Parabolic ACF

Five participants of the workshop chose to work on the project: Mark Allen, Emily Casey, Inwon Kim, Michael Novak, and Mariana Smit Vega Garcia. The goal was to adapt recent results about the elliptic ACF monotonicity formula to the parabolic ACF monotonicity formula. The overall objective of the project consists of three steps.

(1) Obtain a quantitative Faber-Krahn type inequality in the global Gaussian space. Such an inequality was recently proven but with the suboptimal exponent 3. This project will require the optimal exponent 2.
(2) Use the quantitative Faber-Krahn type inequality prove a quantitative parabolic ACF monotonicity formula with a quadratic remainder.
(3) Use the quantitative parabolic ACF formula to obtain a rectifiability result for the set of points where the parabolic ACF is positive. This would give an immediate rectifiability result for the two-phase parabolic Bernoulli free boundary problem.

During the workshop the group focused on step (2). The parabolic ACF monotonicity formula lacks any control of the time derivative which presents a serious challenge to adapting the proof for the elliptic ACF to the parabolic ACF. During the focused sessions of the workshop, the group discovered new insights which allowed us to overcome this challenge. Four of the participants in the group committed to continue to work on the problem after the workshop. The group is currently writing up the proof to step (2). The group will then work on steps (1) and (3).

ϵ-regularity near two-plane points

During the workshop, we considered several rigidity/resolution questions regarding the one-phase Bernoulli problem. In particular, we were interested in possible epsilon regularity results nearby the solution $|x_n|$. We proved two initial rigidity results (one in all dimensions, one in the plane) for such solutions under some additional natural hypotheses during the week. These serve as a jumping off point for more difficult questions of ϵ regularity/generic smoothness/local finiteness of branch points in 2D/etc. We intend to work on these questions, and have organized a group scheduling method and communal Tex file containing the theorems we proved in the conference and references for future investigations.

Gradient flows related to the Faber-Krahn Inequality

At the workshop, our group worked on defining a gradient flow corresponding to eigenvalue minimization problems, focusing on the most basic case of the first Dirichlet eigenvalue in the close-to-spherical setting. We discussed several different approaches, including a JKO-type discrete time scheme, and some complex-variable approaches in the plane. As a warm-up we focused on the more basic problem of a flow that disregards the volume constraint and expands in a way that optimally decreases the eigenvalue. One can then ask whether a rescaling of the solution converges to a circle. We computed an explicit solution, which is a sphere that expands at a precise rate—this serves as the analogue of the equilibrium solution. Following the workshop, the group exchanged some emails about different ideas and approaches. We are not currently working on the problem, but agreed that there is interest in coming back to it at some later date.

Sliding Drops

This group was interested in a free boundary problem related to liquid droplets sliding down an inclined plane. Of particular interest is the geometry of the tail. Physical experiments show that, as the volume of the drop is increased, there appears a corner or cusp at the tail, and for even larger volumes the corner smooths out and the travelling droplet meets the surface at zero contact angle to a post-wetted surface. One simplified model of this scenario involves a contact line motion problem of Bernoulli-type. Unfortunately the full volume constrained travelling wave problem has proved extremely challenging so the goal of this project was to devise a simpler model where the issue of tail geometry could be explored.

After the first day we settled on a natural simplified model without volume constraint, wherein we were able to predict that the tail-shape of the droplet could only be a corner of a particular opening angle determined exactly by the wave-speed c (in particular the cusp was ruled out). We also considered the limit $c \to \infty$ where we derived another simple model displaying the transition from Bernoulli-type to obstacle-type free boundary condition. We also proved several useful technical Lemmas on the solution which convinced us that these heuristic ideas could be made rigorous. The group members have now met several times via Zoom in order to complete the project and write up the results.

Flatness implies smoothness for $V = |Du|^2 - 1$

This model is a variation of the Hele–Shaw problem. Consider a time-parameterized family of non-negative harmonic functions $u = u(x, t)$ such that the boundary of its positivity set moves with prescribed normal speed V. When the speed is a positive constant, one recovers the classical Hele–Shaw problem, which has been extensively studied in the literature. By contrast, the case in which the speed depends on the gradient of the function, allowing both forward and backward motion, remains open. The goal of this project was to understand the regularity of the free boundary of solutions under a flatness hypothesis.

During our stay at AIMS, the group was able to accomplish the following tasks:

Identify the planar fronts of the problem around which we intended to perform the perturbation analysis. Establish compactness of solutions around those planar fronts. Identify the blow-up problem. A major challenge in this problem is the lack of continuity of the solutions in time. This continuity could be imposed on the initial planar front; however, it seems that the iterative argument requires correcting the profile by means of a discontinuous front.