# The Bochner technique

organized by

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Workshop Summary

## Introduction

The workshop focused on recent developments and new applications of Bochner-type methods. The classical Bochner technique combines Weitzenböck formulas with curvature positivity to obtain vanishing, rigidity, and classification results. A recurring theme of the week was that this strategy remains highly effective, but its modern applications often require a refined choice of bundle, curvature term, and algebraic positivity condition.

Discussions centered on several interrelated directions, including the curvature operator, the curvature operator of the second kind, the Calabi curvature operator in Kähler geometry, nonlinear Kodaira–Bochner formulas, complex sectional curvature, quaternion-Kähler geometry, and questions related to geometric flows. Across these topics, participants considered how analytic identities interact with representation theory, special geometric structures, and sharp curvature conditions.

This workshop has led to joint research projects, novel theoretical insights, and broadened applications of the Bochner technique within complex and Riemannian geometry. We are optimistic that the workshop will catalyze further developments and fruitful collaborations in these directions.

## Group Projects

In this section, we record the main projects developed during the workshop and the progress made by the working groups. Some projects led to explicit computations and concrete routes toward new theorems, while others clarified obstructions, reformulated open problems, or identified promising directions for continued collaboration.

*Ricci Flow and Curvature Operator of the Second Kind.*
Our group investigated whether positivity conditions for the curvature operator of the second kind (and the Calabi curvature operator) are preserved under the Ricci flow. Motivated by the work of Lei Ni and Bettiol-Krishnan, we considered a manifold with cohomogeneity-one symmetry in dimension four. The sectional curvature and, consequently, the full curvature tensor (and each component: Weyl tensor, Ricci curvature, scalar curvature) could be computed explicitly using parameters of the metric (functions of one variable). Furthermore, with this setup, the Ricci flow reduces to a system of ODEs.

The remaining challenge is to express the positivity condition as inequalities in terms of the same parameters and to study how these inequalities evolve under the resulting ODE system. It is non-trivial as there is little known about the algebra of the curvature operator of the second kind: a priori, it involves a 9x9 matrix, and understanding its eigenvalues is a challenge. With assistance from AI tools (ChatGPT Plus and NotebookLM), we discovered a useful algebraic decomposition. That is, after choosing a special orthonormal basis for traceless symmetric 4 matrices, the $9 \times 9$ matrix representing the curvature of the second kind consists of block diagonal matrices of sizes $3 \times 3$ (one) and $2 \times 2$ (three blocks). Eigenvalues of these block matrices can be computed relatively easily in terms of metric parameters.

Based on our current analysis, constructing a counterexample appears to be the more promising direction. What remains to be done:

- Examine how inequalities, representing that the curvature of the second kind being positive, evolve along the ODE system.
- To construct a closed manifold, it is sufficient to describe boundary conditions for the metric parameters. We need to make sure that it is consistent with the aforementioned inequalities.
- Obtain estimates to show that within a short time of evolution, the aforementioned inequalities no longer hold true.

*Quaternion-Kähler Manifolds and the LeBrun-Salamon conjecture.*
The quaternionic Kähler group focused on understanding the LeBrun-Salamon conjecture, which asserts that a compact quaternionic-Kähler manifold of positive scalar curvature is isometric to a symmetric space, with the intention to read past work on the subject and consider possibilities for future progress.

Let $B$ be a quaternionic manifold of dimension $4n$. Then the endomorphism bundle $End(TM)$ admits a 3-dimensional parallel subbundle locally spanned by almost complex structures $I, J, K$ which satisfy the quaternionic relations. This gives rise to an associated $SO(3)$-principal bundle $M$ over $B$, called Konishi bundle, which admits a 3-Sasakian structure.

The work of Brendle-Semmelmann suggests to consider a Bochner formula restricted to quaternionic lines spanned by $\{X, JX, IX, KX\}$, for $X \in T_pB$. The goal is to formulate this as a Bochner formula on $M$ with respect to a connection that preserves the metric and the Sasakian structure on the total space, with torsion additionally parallel with respect to the connection. This is the setup for trying to apply the circle of ideas from Moroianu-Semmelmann-Weingart (arXiv:2412.00385). The hope is that when considering Bochner formulas on the $SO(3)$-bundle, we can fully manipulate the quaternionic relations of $I, J, K$.

We started with a general Laplacian calculation of the curvature tensor $R$, referring to Brendle-Semmelmann (arXiv:2506.22181). The objective is to use the quaternionic condition and the curvature assumptions to recognize curvature terms, which in turn could shape any additional assumptions we may need to make.

*A Tachibana Theorem for the Calabi Curvature Operator.*
Our problem was inspired by the possibility of achieving a Calabi curvature operator analogue of Tachibana's theorem. That is, whether a Kähler-Einstein manifold whose Calabi curvature operator is sufficiently $p$-positive necessarily implies local symmetry. Toward this goal, we sought to calculate the Calabi operator on $CP^k \times CP^{n-k}$ via strategies that were previously studied by Petersen and Wink. We were successful in the sense that a Bochner formula could be developed. Applications are subject to continued work after the workshop.

*Alternate Proof of a Result of Bourguignon.*
In our working session, we studied the following theorem: let $(M, g)$ be a four-dimensional Riemannian manifold with harmonic curvature tensor and nonzero signature; then $(M, g)$ is

Einstein. The study was motivated by a seminal paper of Bourguignon, whose proof relies, in his own words, on the "magic" of the Bochner formula. There was a consensus among participants that the proof is not very illuminating, and that a different perspective on the problem would be valuable.

We first focused on understanding the original proof in Bourguignon's paper and the mechanism behind it. We found that the use of Bochner formulas is somewhat incidental: the proof seems to rely less on the harmonicity of the full curvature tensor itself, and more on the fact that the Ricci tensor is Codazzi. By placing the argument in a more algebraic framework, we then tried to extend it to related settings, namely Bach-flat metrics and metrics with weighted harmonic curvature. The Bach-flat case did not seem to yield a useful extension of the theorem, while the weighted harmonic curvature case appeared more promising. In that setting, we eventually found that the argument carries over to show that $(M, g)$ is a Ricci soliton. However, this extension does not require fundamentally new ideas; it is more an observation about the structure of the original proof than a new argument.

With a better understanding of the proof, we then suggested that it may be possible to identify directions for extending the approach to higher dimensions, beginning with dimension 8, though this remains to be pursued after the conclusion of the workshop.

*Complex Sectional Curvature and* $\mathrm{Ric}_L(\omega)$*.*
The purpose of this group discussion is to answer the following broad question:

> *Can one apply the Bochner technique to complex sectional curvature constraints?*

To yield cohomological results via the Bochner technique, one analyses the forms that lie in the kernel of the *Lichnerowicz Laplacian.* The curvature term arising from this Laplacian is known as the *Weitzenböck curvature operator*, denoted $\mathrm{Ric}_L$. In this language, the goal is to identify complex sectional curvature in $\mathrm{Ric}_L(\omega)$ for some form $\omega$.

For special cases, complex sectional curvature may be manifested as three different curvature notions; *sectional curvature*, *isotropic curvature*, and a sum of two sectional curvatures also known as *second Ricci curvature.* In the case of 2-forms, Micallef–Wang (Duke Math Journal, 1993) studied the relation with isotropic curvature, and in the textbook by Petersen the work has been generalized to a special case of $(0, 2)$-tensors in relation to complex sectional curvature.

The challenge for a general $k$-form is that it does not admit the symmetries and decompositions appearing in $(0, 2)$-tensors, so one must rely on a new technique to identify complex sectional curvature for $k > 2$. The group first computed $\mathrm{Ric}_L(\omega)$ for a *simple $k$-form* $\omega$. While the calculation is lengthy, we observed sums of various sectional curvature terms in some preliminary low dimension calculations. The hope is to see curvature patterns in $\mathrm{Ric}_L(\omega)$ in higher dimensional and more general settings, possibly corresponding to one of the three types mentioned above, and apply further techniques such as those found in Ni (Crelle, 2021) to formulate conditions for complex sectional curvature.

*Sectional Curvature and* $K(R, \mathrm{Sym}^{p\cdot})$
Our group studied the curvature term in the Bochner formula on symmetric $p$-tensors, denoted during the workshop by $K(R, \mathrm{Sym}^p)$. It is known that nonnegative sectional curvature implies that the curvature term is nonnegative definite ($K(R, \mathrm{Sym}^p) \succeq 0$). (Indeed, it has been shown that nonnegative sectional curvature is equivalent to the condition that $K(R, \mathrm{Sym}^p) \succeq 0$ for all $p \geq 1$.)

We investigated whether this result could be strengthened to find a weaker condition which would guarantee $K(R, \mathrm{Sym}^p) \succeq 0$ for a fixed $p$. One conjectured condition is $k-$nonnegative sectional curvature, where $k = \lfloor \frac{n+p-2}{p} \rfloor$: that is, for any orthonormal basis $\{e_i\}$, the sum of the $k$ lowest sectional curvatures among the planes $e_i \wedge e_j$ is nonnegative. The conjecture is true if $k$-nonnegative sectional curvature is replaced by $k$-nonnegative curvature operator.

The conjecture holds for $p = 1$ and $p = 2$ by earlier work. We explored inner products involving integrals and derivatives to leverage the fact that symmetric tensors can be represented by homogeneous polynomials. We succeeded in decomposing $\langle K(R, \mathrm{Sym}^p)\phi, \phi\rangle$, where $\phi$ is a symmetric $p$-tensor, into a linear combination of sectional curvatures with coefficients depending on $\phi$. Since the coefficients were not guaranteed to be nonnegative for $p \geq 3$, we have not yet found a way to leverage $k$-positivity of sectional curvature.

*Does* $\mathrm{Ric}^{\perp} < 0$ *imply projective?.*

**Question:** Suppose $(X, g)$ is a compact Kähler manifold with negative orthogonal Ricci curvature. Is $X$ projective?

Several approaches were considered. One strategy was to adapt methods in the spirit of Wenhao Ou, beginning from the assumption of non-projectivity to produce a nontrivial holomorphic $(2, 0)$-form. The idea is that the kernel of such a form is a non-trivial distribution, and establishing suitable stability properties could then lead to a contradiction under the curvature hypothesis.

A second strategy was to adapt the Wu–Yau and Tosatti–Yang approach to obtaining a Kähler–Einstein metric under curvature conditions involving orthogonal Ricci curvature. This would then imply projectivity via the Kodaira Embedding Theorem, provided one can obtain the necessary analytic control. The main technical obstruction is the establishment of uniform $C^2$-estimates for solutions to the associated complex Monge–Ampère equations.

Partial progress was obtained in complex dimension two, where algebraic identities arising from polarization arguments were obtained under the assumption of negative orthogonal Ricci curvature. It remains to be seen whether these identities can be refined into effective curvature control sufficient to close the required $C^2$-estimates.

## *Future Plans*

The organizers were extremely pleased with the outcome of the workshop, which exceeded our expectations in both its mathematical depth and its collaborative energy. The week provided participants with a valuable opportunity to learn from one another, clarify important open problems, and make tangible progress on several active research directions related to the Bochner technique. The workshop also initiated several new collaborations, strengthened existing ones, and produced research projects that are expected to develop into joint publications.

Going forward, the organizers and participants plan to continue the group projects initiated at AIM through regular follow-up discussions, shared notes and computations, and the preparation of research papers based on the progress made during the workshop. We also hope that the workshop will serve as a foundation for further meetings and collaborations in this area.

The organizers are deeply grateful to the AIM staff for their extraordinary support, hospitality, and encouragement, all of which contributed greatly to the success of this workshop.