Buildings and combinatorial representation theory

organized by
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Workshop Summary

Goals. Bruhat–Tits *buildings* play an essential, not yet well-understood role in *combinatorial representation theory* by providing a geometric realization to existing combinatorial models and linking them to the algebro-geometric tools of representation theory. In the past 5 years, connections have been made between buildings and representation theory and objects in other fields like tropical geometry, and (non-Archimedean) amoebas. In each area, mathematicians have been developing their own sets of tools to attack the related representation theory. This AIM workshop brought researchers from these areas together to see how those tools fit together, and see what range of problems we can solve by combining these tools.

If we think of placing

(1) representation theory
(2) buildings
(3) tropical geometry

at the vertices of a triangle, one goal of the workshop was to move along the edges of this triangle.

Participants. To achieve our goals, we brought together experts from the different fields to discuss recent results, to share their tools, and to compile a list of open problems and conjectures. At least one person from each major area was asked to speak, and additionally, we asked some participants to lead small afternoon tutorials on the "Basics of" key concepts. This was needed, as not every participant was familiar with the different combinatorial models, or the underlying algebraic geometry or tropical geometry.

This group of mathematicians consisted of combinatorialists, representation theorists, geometers, algebraic geometers, tropical geometers, and computationally oriented mathematicians. Among the participants, we also included a number of graduate students, recent graduates, and mathematicians from primarily teaching colleges.

Structure. The structure of an ARCC workshop was very well-suited to our objectives and needs, and toward fostering collaboration. We arranged for a small number of hour-long survey lectures in the first few mornings, and a few more specialized lectures later in the workshop. On the first day, during the afternoon discussion, we found that in addition to having working groups and open problem sessions, we needed several small tutorials bringing people up to speed on the vocabulary and ideas surrounding some key concepts, such as buildings, galleries, LS paths, MV cycles, MV polytopes, and crystals. Finally, we allowed for several large blocks of time for small (and large) group meetings. During this time, some small groups started to work on some of the questions raised during the discussion sessions, while others used the time to continue already existing collaborations. In general, the fairly

open schedule encouraged many useful discussions and paved the way for new projects and collaborations.

The conference opened with a mixture of survey lectures and short talks that introduced the different piecewise-linear combinatorial models that arise in representation theory, the problems they have been used to solve, and the open problems that remain.

A few more lectures followed throughout the week. In addition, during the discussion sessions, several topics for working groups to attack were listed, and were discussed during the small group meetings. There were roughly five different groups (that changed daily) that consisted of from two to ten members, some of the smaller ones were continuing collaborations and others were starting new collaborations. One afternoon, Arun Ram met with several of the junior researchers in a group to find out what they had been working on and thinking about, and to give them some possible future directions. At the close of the workshop, we had a wrap-up session, where groups informally presented the progress they made on their problems.

Talks. We had two overview lectures the first day. Arun Ram gave a survey that discussed the relationship between the loop Grassmannian, Mirkovic-Vilonen (MV) cycles, the moment map, spherical Hecke algebras, the Satake correspondence, and our combinatorial understanding of all of these. Misha Kapovich discussed triangles, structure constants, and tensor product multiplicities, and how Euclidean buildings entered the picture. He also listed several open problems and conjectures throughout his talk. These problems were revisited in several working groups throughout the workshop. Allen Knutson gave a brief (15 minutes) overview of the connection to structure constants in Schubert calculus.

In the afternoon organizational session, we found several participants wanted to get a firmer grasp on the "Basics of buildings," "Basics of LS paths and MV cycles." We split up into small groups, some into these tutorials, others into working groups on open problems. When we re-convened, with an eye toward the Tuesday morning topics, Bernd Sturmfels led a group exercise (or "competition") on tropical geometry. In small teams, we raced to answer questions and work out examples in tropical linear algebra, and to describe tropical triangles. It was an engaging hands-on exercise that got neighbors communicating, and really broke the ice.

We opened the second day with a lecture by Arkady Berenstein on Polytopal Models and Tropical Geometry. Jenia Tevelev talked about Amoebas in Buildings, Tropicalization and Compactification. In the afternoon, we had more "Basics of" groups, in addition to working groups. We also had some short "Preview" talks–these were introductions to topics to see if participants wanted to form a related working group.

On day 3, we had morning talks by Stephane Gaussent on the Littelmann path model, LS paths and their crystal structure, MV cycles. Joel Kamnitzer talked on geometric Satake, hives and triangles in the loop Grassmannian. The afternoon was spent in working groups.

On day 4, we continued to have some morning talks, this time by David Nadler on Generalizing Geometric Satake and categorification. Allen Knutson described honeycombs and how they were used to prove saturation. Again and again, we were seeing the importance of convexity, tropicalization, and the piecewise linear combinatorial models that arise when systems of inequalities are imposed. We also had a group discussion reviewing what had been accomplished to date, and what problems and directions we needed to move toward. Ulrich

Goertz gave a "Preview" talk on Shimura varieties and affine flag varieties. The afternoon was devoted to working groups.

On the final day, we had one morning talk by Misha Kapovich on saturation. Then we broke into working groups, this time divided by leader rather than topic, as we found on the previous day everyone wanted to clump together on the same problem, but inevitably broke into smaller groups anyhow. This new strategy worked well. We had a wrap-up session at the end to discuss progress and reports from the working groups. (We also took a short break to listen to an NPR broadcast about recent developments regarding E_8, which included some sound bites by AIM participants.)

Group work and Collaborations. We formed small groups to work on a subset of open problems that were discussed in Misha Kapovich's opening talk and that were also listed daily on the board, and the small groups continued to meet throughout the workshop.

On the first day, while some groups were in tutorials, a working group started discussing what buildings for general Kac-Moody groups are or should be.

After Arun Ram gave a "Preview" talk on buildings for noncrystallographic Coxeter groups, a working group began discussing this topic. What is the associated building for complex reflection groups? Arun Ram suggested recent work on p-compact groups gave the answer. Significant discussions took place on the possibility of using a path description of the related classifying space as a method of realizing this space as a space of "paths in the building".

Further work on this from a slightly different viewpoint was investigated by Misha Kapovich with Arkady Berenstein. They started with the related system of inequalities, which exists even in the absence of the obvious analogues of the spaces that exist for Weyl groups. This was the formation of a new collaboration.

One small group was led by Christian Lenart and Sophie Morier-Genoud. They discussed different models of crystals and properties of canonical bases. Christian Lenart also introduced his computer software, and was available to help other participants install this software on their computers and give tutorials in using it.

A particularly vibrant group centered around Joel Kamnitzer, James Parkinson, and Jacqui Ramagge. They started with a conjecture for a combinatorial bijection from Littelmann paths to MV polytopes, which used the idea of taking a convex hull of retractions in the sectors corresponding to W. However, from some examples, we saw their combinatorial picture of retraction was incomplete. Jacqui Ramagge conjectured that to capture retraction, one needed to fold an entire region (a convex hull) of possible paths. This group continued working long past the day's end, and this new collaboration is continuing.

Another group talked about formulating an explicit correspondence between triangles and hives.

Misha Kapovich mentioned that after discussion with Kamnitzer and Knutson, he understood the combinatorial differences between MV polytopes in the simply laced versus non-simply laced case.

A continuing collaboration between Tom Haines and Ulrich Goertz studied affine Deligne-Lusztig varieties. Their group reported some progress toward interpolating between I orbits and U orbits.

Some ongoing collaborations between Belkale, Kapovich, Kumar, Millson continued.

Open Problems. There were more problems posed than the groups had time to work on. Some of the open problems (which will also be listed on the website) and partial solutions include:

- Find the MV polytope of an MV cycle from the Littelmann path data. As mentioned above, a new collaboration formed around this problem, and they have a conjectured solution they are testing.
- More generally: understand the different combinatorial models involved (such as Knutson-Tao honeycombs, MV polytopes, Littelmann path models, canonical bases), provide a dictionary between them, and lay the groundwork to enable researchers to apply these tools toward a host of related problems. Substantial progress was made toward pieces of this during the workshop.
- Give an analogue of buildings for complex reflection groups. Arun Ram proposed a solution of what the building should be. The theory of p-compact groups says that there is a "p-compact group" X corresponding to each $_p$-reflection group and that this p-compact group X contains a maximal torus T. The "quotient" X/T is an analogue of the flag variety corresponding to the reflection group. This is a starting point, but that does not mean we know what the building looks like or how to extract information from it. Many directions and open problems follow.
- Give an analogue of buildings for noncrystallographic reflection groups. In the case of a noncrystallographic reflection group, it is known how to construct the graded Hecke algebra. The graded Hecke algebra is a "degeneration" of the affine Hecke algebra which contains most of the information of the affine Hecke algebra. One can write a Satake map for the graded Hecke algebra and use this to compute numbers of triangles in the corresponding building, even though one does not know a construction of this building. There are inequalities which on can read off the Schubert calculus.
- Make an explicit correspondence between triangles and hives. Joel Kamnitzer explained a beautiful proposed solution to this problem using an action of three copies of the affine Grassmannian on three wedge powers, and a "fake moment map" on these triples.
- Understand to what extent the inclusions $C_{\text{Hecke}} \subseteq C_{\text{Rep}} \subseteq C_{\text{Tri}}$ are strict, where C_{Hecke} is the cone of structure constants of the spherical Hecke algebra, C_{Rep} is the cone of tensor product multiplicities and C_{Tri} is the cone of triangles in the building.
- Examine and compare the different approaches to the saturation theorem, with an emphasis on the role of buildings, to get more precise answers (in all types) and improve the proofs, and possibly also make a sensible Horn conjecture in other types.
- Find an analogue of hives outside the type A case.
- What is the analogue of stretching of paths on the level of points in the affine flag variety or the loop Grassmannian?
- Compute when general affine Deligne-Lusztig varieties in the affine flag variety are nonempty and, if possible, compute their dimension.
- In what way can the combinatorics of path models and MV cycles be applied to the Langlands program? David Nadler explained to us that, from the point of view of the geometric Langlands program, the geometric Satake correspondence ought to be lifted to an equivalence of categories.

Outlook. We hope that significant progress will be made on the open problems and conjectures discussed at AIM, and will be continued in the collaborations begun at the workshop. We found the idea of working groups and of having very few lectures to be very useful for fostering discussion and collaboration.

The organizers received positive and enthusiastic feedback from several participants. One, who came from a small teaching college, emailed us: "Thank you for organizing a wonderful workshop. Even though many of the ideas discussed at the workshop are not directly related to my research area, I have benefited greatly from attending. In particular, I learned much about the different combinatorial models used in representation theory and the existing and conjectural relations between them. I will start exploring some of these models and possible connections with the p-adic world. I found the format of the workshop very efficient for developing possible collaborations as well as for deeper exploration of the ideas presented in the morning talks."

In conclusion, the ARCC workshop on Buildings and Combinatorial Representation Theory was a definite success. Everyone participated in at least one small group session, and we expect many of these sessions to lead to longer-term collaborations and substantial progress in the field. We owe a large debt of gratitude to the AIM staff for organizing and facilitating the workshop and to AIM and the NSF for funding the conference.