Categorified Hecke algebras, link homology, and Hilbert schemes

organized by
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Workshop Summary

The workshop brought together experts with a wide range of expertise, from categorification and geometric representation theory to low-dimensional topology, algebraic geometry and combinatorics. They exchanged their experience, found common points of interest and actively engaged in the working groups in the afternoon. There were several highlights of the workshop, such as the lecture of Ben Elias outlying many of the ideas and constructions pertaining to the categorification of the affine Hecke algebra.

The workshop led to new collaborations which are likely to bring new perspectives to categorification of finite and affine Hecke algebras, matrix factorizations, affine Springer fibers and Hilbert schemes of points. Some participants described their experience as "the most intense workshop they've ever been to", which we regard as a sign of successful and fruitful interaction.

Morning talks

Every morning there were two talks on various topics. As instructed by the workshop organizers, the speakers kept the talks at introductory level to make them accessible to all participants.

Oblomkov-Rozansky theory.

About 10 years ago, Khovanov and Rozansky introduced a link homology theory categorifying HOMFLY-PT polynomial, a construction which Khovanov later rephrased in terms of Soergel bimodules. More recently, Oblomkov and Rozansky introduced another theory using matrix factorizations on a certain space built out of a semisimple Lie group, its Borel subgroup and nilpotent radical. The Oblomkov-Rozansky theory also categorifies the HOMFLY-PT polynomial, and it is natural to expect that the two theories agree or are very close to each other, despite their very different constructions.

Alexei Oblomkov and Lev Rozansky outlined their theory in their talks: the former gave details on the construction of the category of matrix factorizations, and the latter showed how many of their expectations stem from quantum field theories. Tina Kanstrup described her work with Arkhipov, relating the matrix factorizations considered by Oblomkov-Rozansky to some other spaces more familiar in geometric representation theory (such as Springer resolution or Steinberg variety). She also outlined a proof of the equivalence between the affine braid group action defined by Oblomkov-Rozansky using matrix factorizations and the affine braid group action that Bezrukavnikov-Riche define by using Steinberg variety.

Soergel bimodules and Khovanov-Rozansky homology.

Matthew Hogancamp discussed the general constructions of Soergel bimodules, their Hochschild homologies and Khovanov-Rozansky homology. He also reviewed recent progress in the computations of Khovanov-Rozansky homology, achieved by himself, Elias and Mellit. In particular, he explained that the Khovanov-Rozansky homology of positive torus links have the "parity property", that is, are concentrated in even homological degrees. This is very important for the purpose of actually computing link invariants.

Affine Soergel bimodules and flattening.

Ben Elias described his work in progress on the categorification of the extended affine Hecke algebra, and of its projection to the finite Hecke algebra ("the flattening functor"). This construction uses a certain extension of the affine Soergel category introduced by Mackaay and Thiel. Elias constructed a family of objects in the Drinfeld center of the affine Hecke category, and explained that they are filtered by "Wakimoto objects" corresponding to the lattice part of the affine Weyl group. After projecting to the finite Hecke category, the central objects are expected to stay central, and "Wakimoto objects" become Rouquier complexes for twist braids. Meanwhile, Kostiantyn Tolmachov defined a geometric version of the same construction using his work with Bezrukavnikov and others. In particular, he related the center of the finite Hecke category to character sheaves.

Skein theory and generalizations.

Several talks were focused on skein theory for 3-manifolds and surfaces, its generalizations and possible categorifications: Peter Samuelson reviewed the construction of the skein algebra of a surface, skein module for a 3-manifold, and explained his work with Hugh Morton giving the explicit description of the skein algebra of a torus. He also related it to the specialization of the elliptic Hall algebra studied in geometric representation theory. David Jordan put the skein construction in the broader categorical context, and discussed various topological quantum field theories related to quantum groups. Anton Mellit discussed an interesting algebra $\mathbb{A}_{q,t}$ appearing in his proof (with Erik Carlsson) of the Shuffle Conjecture in algebraic combinatorics, and its connection to skein theory. Paul Wedrich discussed his work with Hoel Queffelec on categorification of $\mathfrak{sl}_2$ skein algebras for surfaces using webs and foams.

Problem list

On Monday afternoon, the participants have suggested over 20 problems related to the topics of the workshop. Ben Elias served as a moderator of the discussion and recorded all the problems on the whiteboard while Tina Kanstrup typed them in a list on her computer. The list is available at

http://aimpl.org/catheckehilbert/

Working groups

Comparison between Oblomkov-Rozansky and Khovanov-Rozansky invariants.

This problem has attracted a lot of attention. The participants discussed various technical details of the Oblomkov-Rozansky theory and arrived at a promising conjecture relating this theory to the diagrammatic formalism of Elias and Williamson. In more detail, the goal is to construct an equivalence:

$$K^b(\mathrm{SBim}_n) \overset{\cong}{\leftrightarrows} MF_{B\times B}(\mathfrak{b} \times G \times \mathfrak{n}, W)$$

where $G = GL_n$, B is its Borel subgroup, $\mathfrak{b}$ is the Lie algebra of upper triangular matrices, and $\mathfrak{n}$ is the Lie algebra of strictly lower triangular matrices. In order to construct the functor $\rightarrow$, one must construct objects in the category MF corresponding to the Bott-Samelson generators $\{B_i\}_{1\leq i<n} \in \mathrm{SBim}_n$ (which has already been achieved by Oblomkov-Rozansky) and various morphisms between tensor products of these objects that must be checked to hold, up to homotopy, in the category MF (this remains to be checked, but we believe that it boils down to a computation involving small values of n, such as $n = 2, 3$). For the opposite functor $\leftarrow$, we expect that it arises naturally from the 2-category that featured in Lev Rozansky's talk.

The group also discussed ways to relate the setting of the conjectures from the work of Gorsky, Negut and Rasmussen to the methods of Oblomkov-Rozansky and Arkhipov-Kanstrup. In particular, it was explained that categorical Chern character of Oblomkov-Rozansky is a particular example of the partial braid closure functor that is constructed in the work of Gorsky, Negut and Rasmussen and thus constructing analogous partial braid closure functor in the matrix factorization setting could get two approaches closer to each other.

Higher Hochschild homology via tautological objects.

In the definition of Khovanov-Rozansky homology, higher a-degrees are encoded by the higher Hochschild homology of Soergel bimodules. However, this operation is not very natural from the categorical viewpoint, and it is conjectured that higher Hochschild homology are representable by some central objects in Soergel category. Specifically, it is expected that

$$HHH^i(B) = \mathrm{Hom}(E_i, B)$$

for all Soergel bimodules B and certain specific central objects E_i. During the workshop, Anton Mellit, Matt Hogancamp and Eugene Gorsky proved this conjecture for the top Hochschild homology (with E_n being the inverse full twist), generalizing an earlier observation of Keita Nakagane.

Homology of Springer fibers and Jones-Ocneanu traces outside of type A.

It follows from the work of Gorsky, Oblomkov, Rasmussen, Shende and Yun that in type A the homology of certain affine Springer fibres are closely related to the Jones-Ocneanu traces of the corresponding conjugacy classes of braids. It was unclear if such results extend to other types, but Minh-Tam Trinh answered this question positively in many cases. He presented his results in this working group, and the participants worked on extending them to more general cases.

Minh-Tam Trinh presented to the rest of the group general shape of his conjectures which are concerned with the analogs of torus knots in other types. It was explained in the work of Oblomkov and Yun that other type torus braids are the compositions of the braids of the regular (in sense of Springer) elements of the Weyl group (embedded inside the corresponding braid group) with some power of the full twist. The corresponding homogeneous Affine Springer fibers were also studied in details by Oblomkov and Yun, they constructed the action of the rational Cherednik algebra on the cohomology of the corresponding Springer fiber, but the cohomology is not a simple module over the algebra and that is where a lot subtleties originate.

The initial conjecture of Minh-Tam is concerned with comparing of the Poincare polynomial of the corresponding Affine Springer fiber and the value of the Jones-Ocneanu trace on the torus braid. He discovered that the naive conjecture does not work because the Jones-Ocneanu trace for the small order regular elements of the Weil group have Laurent expansion that is not positive (the worst element of this sort is -1 which is regular in all types outside type A). By looking at examples participants of the group found an explanation for the negative coefficients in the expansion, they are related to the odd cohomology of the corresponding homogeneous Affine Springer fiber. Thus the conjecture was fixed and now works in all homogeneous examples that are computable by hands.

Drinfeld center of the category of Soergel bimodules.

It is well known that the center of the finite Hecke algebra is generated by symmetric functions in q-deformed Jucys-Murphy elements, and the center of the affine Hecke algebra surjects onto it. The conjectures of Gorsky, Negut and Rasmussen predict a relation between the categorification of this center and the derived category of coherent sheaves on the Hilbert scheme of points on the plane. In particular, the tautological bundle T on the Hilbert scheme should categorify the sum of Jucys-Murphy elements, and its exterior powers $\wedge^i(T)$ should categorify elementary symmetric functions and correspond to central objects E_i above. It is expected that the analogue of T has two commuting endomoprhisms X and Y.

During the workshop, the participants discussed various approaches to the construction of central objects, including flattening functors of Elias and Tolmachov, categorical Chern character of Oblomkov and Rozansky, and y-ified partial trace of Gorsky and Hogancamp. In the first two approaches the resulting object clearly belongs to the Drinfeld center, but its endomorphisms are yet unclear. In the third approach the action of X and Y is manifest, but the proof that the object is in the Drinfeld center is not clear.

The participants also discussed the structure of the Drinfeld center as a braided monoidal category, and of a subcategory generated by T. In particular, they conjectured that the braiding restricted to T is symmeric, and Tolmachov presented some geometric arguments explaining this in his geometric approach.

Combinatorial formulas for Jucys-Murphy braids and Demazure crystals.

It was observed by Gorsky, Negut, Oblomkov, Rozansky and others that the Jucys-Murphy braids (twisted by the Coxeter braid) should correspond to explicit line bundles on the flag Hilbert scheme. The equivariant Euler characteristics can be computed by localization at fixed points, and yield explicit combinatorial sums over standard Young tableaux. Anne Schilling observed that these sums specialize at $t = 1$ to characters of certain Demazure crystals, and asked if the full Euler characteristic depending on q and t has a combinatorial or crystal-theoretic interpretation. In special cases, this recovers q, t-Catalan polynomials and their rational generalizations.

In a working group with Julianne Rainbolt and Nicolle Gonzales, Schilling made a significant progress on this problem. They wrote a Sage code computing these q, t-polynomials and computed them explicitly. For Jucys-Murphy braids up to four strands, they conjectured explicit recursive relations for such polynomials and tested them in many examples. It is likely that these recursions can be generalized to higher number of strands and lead to general combinatorial formulas.

Future plans

The workshop significantly strengthened the interaction between the various groups of researchers. Many connections between various approaches to categorification of Hecke algebras and their centers were discussed and compared. We expect that the progress achieved in working groups will lead to new collaborations and joint papers in the near future.

Also, this workhop will be followed by other workhops and activities on similar or closely related topics. Elias, Gorsky, Negut, Oblomkov, Rozansky and Schilling are co-PIs on NSF Focused Research Group "Algebra and Geometry behind Link Homology" and will organize a workshop at UC Davis in late June 2019, followed by two more workshops in 2020 and 2021. Jordan and Samuelson will organize a workshop on geometric representation theory and low-dimensional topology in Edinburgh in early June 2019. Mellit and Wedrich will organize a conference on categorification in quantum topology and beyond in Vienna in January 2019. Many of speakers and participants of this AIM workshop will speak at these conferences and report on their progress on the above problems.