Generalizations of chip-firing and the critical group

organized by
Lionel Levine, Jeremy Martin, David Perkinson, and James Propp

Workshop Summary

The sandpile/chip-firing model has been studied in many different guises and from many different viewpoints, including algebraic geometry, combinatorics, and probability. The goal of this workshop, which brought together researchers from these diverse areas, was to generalize chip-firing in three directions: (i) replace chips on the vertices of a graph with "flow" on codimension-one faces of a simplicial or cellular complex; (ii) replace vertices with abelian finite automata; (iii) take limits over a sequence of growing graphs.

The schedule mostly followed the AIM model of two expository talks each morning and organized working groups in the afternoon. Monday afternoon was devoted to a moderated problem session led by Vic Reiner, and several subsequent afternoons began with shorter sessions for participants to propose problems. Sam Hopkins served as problem session scribe throughout the week; his detailed list of problems and comments is available from http://www.aimath.org/WWN/chipfiring/aim_chip-firing_problems.pdf.

A major open problem studied at the workshop was the generalization of chip-firing from graphs to cell complexes (#17 in Hopkins' list). Duval, Klivans and Martin had constructed a cellular critical group that agrees with the standard construction for graphs, whose cardinality is a weighted enumerator for cellular spanning trees, and which admits a generalization of the Bacher-de la Harpe-Nagnibeda description in terms of cuts and flows. The hard part is to interpret the elements of the group as critical configurations of a suitable generalization of the chip-firing game. Participants working on this problem included Baker, Duval, Hopkins, Klivans, Manjunath, Martin, Merino, Musiker, and Shokrieh. Baker and Shokrieh's recent work on algebraic potential theory for graphs may provide a promising avenue of attack.

Cartwright's talk on tropical complexes and Picard groups included his conjecture about the tropical Picard group of a triangulation of the product of two graphs. Lazar and Martin got interested in this problem at the workshop, and Lazar (a graduate student) is currently writing Sage code to compute tropical Picard groups. This shows promise of becoming a good student project, and is a potential collaboration that would not have taken place without AIM.

After hearing Sportiello's morning talk, Levine and Propp realized that there may be a link between the "particles and antiparticles" story for rotor routing (due to Levine, Propp and two students) and the "addition operators and subtraction operators" story for chip-firing (due to Sportiello). Conversations at AIM among the three of them provided some support for this hunch, though the real work has yet to be done.

Pegden and Smart, in describing their work on sandpiles in the square lattice, pointed out that the variant theory of F-lattice sandpiles might be a good topic for exploration, since in some respects it is a simplified version of the square-lattice sandpile (there is only one Pegden-Smart cone bounding Γ instead of infinitely many). Sportiello, who has developed his own approach to sandpiles, initially doubted that the cone could tell the whole story for F-lattice sandpiles, but thanks to discussions at the meeting he came to realize that although the apex of the cone doesn't govern what we see everywhere in F-lattice sandpile pictures, paths on the cone might well explain what we see. The work Pegden and Smart did on the F-lattice while at AIM (see Smart's write-up http://www.aimath.org/WWN/chipfiring/flattice.pdf) convinced them that, with some serious work, they could characterize the scaling limit of the sandpile on the F-lattice by recursively constructing a certain family of odometers, much as they did in the case of the ordinary square lattice.

Problem #1, proposed by Backman (generalize chip-firing to k-uniform hypergraphs, with chips represented by kth roots of unity), drew interest. The workshop served to start an ongoing discussion between Backman and Kassel, who is a specialist in the vector bundle Laplacian and hence has experience working with complex-valued Laplacians.

Problem #8, proposed by Ellenberg (when does the set of spanning trees of a graph naturally form a torsor for its critical group?) drew sustained interest for the duration of the workshop, with many people making contributions. Baker suggested that for planar graphs there may be a relation between the rotor router action of the sandpile group on spanning trees of the graph with the action on spanning trees of the dual graph. Chan found a counterexample to Baker's original guess, which led Baker to formulate a slightly more complicated "clockwise/counterclockwise" version of the conjecture; workshop participants (including Baker, Chan, Glass, Macauley, Perkinson, Propp, and Werner) computed concrete examples that corroborated Baker's revised version of his conjecture. (Ellenberg has described the process as "a good example of how AIM is awesome".) The collaboration continues and is likely to produce a very nice result.

Problem #10 (is there a deletion-contraction recurrence for critical groups of graphs?) was considered by a group that included Cartwright, Hopkins, Merino, Perkinson, Reiner, and Shokrieh. The group was led to consider an ideal based on cuts of the graph (related to the toppling ideal) for which reasonable deletion-contraction conjectures could be posited.

Cartwright, Corry, Musiker, Reiner, and Shokrieh reported progress in understanding how the homomorphism of critical groups $K(G') \to K(G)$ associated to a regular covering of graphs $G' \to G$ factors through a covering of real tori $T' \to T$ in which the graphs are embedded.

For Problem #13 (what is the right definition of an abelian network with shared memory?), Diaconis, Pak, Kassel, Huss, and Levine collected examples of abelian processes with shared memory. These fell into several classes: (i) examples such as source-reversal and cycle-popping, which are abelian since legal moves are always disjoint; (ii) herditary chip-firing, which is abelian in a weaker sense (the odometer is well-defined on vertices but not on clusters); (iii) asynchronous sorting and the Olympiad pentagon problem, which are abelian in an even weaker sense (the final state does not depend on the execution but the odometer does). Propp, motivated by a question by Diaconis about disk-packing (Problem #12), came up with a fourth class of abelian processes whose stable states are the antichains of a poset.

Diaconis spoke on a stochastic sandpile model that he had written about with William Fulton. In a working group, Rolla presented some open problems about a stochastic sandpile model in one dimension. After the workshop, Ellenberg found a connection to the random walk generated by two unipotent matrices in $\mathrm{SL}_2(\mathbb{Z})$. He wrote about this on his blog: http://quomodocumque.wordpress.com/2013/07/28/the-stochastic-sandpile/.

Problem #14 (a question of Stanley's on G-parking functions) was solved by Perkinson and his students after the workshop. The solution was aided by conversations at the workshop, particularly with Shokrieh and Pak.