Chromatic homotopy theory and p-adic geometry

organized by

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Workshop Summary

Background

This workshop focused on the interplay between the fields of *homotopy theory* and *arithmetic geometry*. This interplay runs in both directions. For example, homotopy theory borrows substantially from the theory of formal groups, and conversely, some of the objects of p-adic Hodge theory appear naturally as homotopy groups of certain ring spectra. The goal of the workshop was to describe and study the problems in arithmetic geometry that arise from chromatic homotopy theory. A second goal was to foster communication between these two communities of researchers, by way of presenting some recent results of the organizers, including their calculation of $\pi_* L_{K(n)} S^0 \otimes$, the rationalized homotopy groups of the $K(n)$-localized sphere spectrum.

Structure of the workshop

The workshop brought together researchers from a spectrum of career stages (PhD and post-doctoral, all the way up to senior), and also from different subject backgrounds. The organizers attempted to strike a balance between those who primarily study homotopy theory and those who primarily study number theory / arithmetic geometry.

The workshop ran from 9 am to 5 pm, Monday through Friday, although participants often continued to work at AIM until late in the evening. Morning talks were given according to the following schedule:

- 9 am Monday: Paul Goerss, historical background on stable homotopy groups of spheres and Bousfield localization.
- 11 am Monday: Jared Weinstein, crash course in adic spaces and p-adic geometry.
- 9 am Tuesday: Jeremy Hahn, the importance of the moduli stack of formal groups to stable homotopy theory, and the Devinatz-Hopkins spectral sequence.
- 11 am Tuesday: Lucas Mann, perfectoid spaces and pro-étale cohomology of rigid-analytic spaces.
- 9 am Wednesday: Agnes Beaudry, the splitting conjecture.
- 11 am Wednesday: Laurent Fargues, the isomorphism between the Lubin-Tate and Drinfeld towers.
- 9 am Thursday: Tomer Schlank, a crash course in spectra.
- 11 am Thursday: Andrew Salch, computations of the cohomology of the Morava stabilizer group with coefficients in a finite field.
- 9 am Friday: Laurent Fargues, rigid-analytic coordinates on the moduli stack of formal groups completed at height n.

In the afternoons, participants were divided into groups to work on particular problems.

Outcomes of participant groups

Cyclotomic extensions of the $K(n)$-local sphere.
This problem concerns algebraic Picard groups and rational homotopy groups of cyclotomic extensions of the $K(n)$-local sphere. A particular case of interest is the maximal p-cyclotomic extension of the $K(n)$-local sphere, called the $K(n)$-local "half sphere". In their recent breakthroughs, the organizers have computed algebraic Picard groups and rational homotopy groups of the actual $K(n)$-sphere using arithmetic geometry.

During this workshop, we computed that the algebraic Picard group of the $K(n)$-local half sphere is $\mathbb{Z}_p \times \mathbb{Z}/2(p^n - 1)$, at least when $p > 2$. To adapt the organizers' approach in our case, one crucial observation is an equivalence of quotient stacks $[\mathrm{LT}_n//\mathrm{S}\mathbb{G}_n] \simeq [\mathcal{H}^{n-1}//\mathrm{SL}_n(\mathbb{Z}_p)]$ in the Lubin–Tate and Drinfeld two-tower comparison. Here, $\mathrm{S}\mathbb{G}_n$ is the kernel of the (unreduced) determinant map $\det\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$. This observation should also allow us to compute algebraic Picard groups of finite cyclotomic extensions of the $K(n)$-local sphere, which we plan to work on after the workshop.

Chromatic vanishing in degree 1.
Let W denote the ring $W(\overline{\mathbb{F}}_p)$ of p-typical Witt vectors of the algebraic closure of $\mathbb{F}_p$, let $A_n \cong W[\![u_1, \ldots, u_{n-1}]\!]$ denote the Lubin-Tate ring, and let $\mathbb{G}_n$ denote the Morava stabilizer group. The goal of this project is to show that $H^1(\mathbb{G}_n, W) \to H^1(\mathbb{G}_n, A_n)$ is an isomorphism. This is a special case of the chromatic vanishing conjecture, which asks for the map $H^*(\mathbb{G}_n, W) \to H^*(\mathbb{G}_n, A_n)$ is an equivalence. Previous work has shown that this holds when $n = 1, 2$ or when $* = 0$. From the work of Barthel, Schlank, Stapleton, and Weinstein, we know this is also true rationally.

During the workshop we showed that, for $* = 1$, the map in question is in fact the kernel of a map $b : H^1(\mathbb{G}_n, A_n) \to H^1_{\text{pro-ét}}(\mathcal{H}_C^{n-1}; \mathcal{O}^+)^{\Pi_n}$, where $\mathcal{H}_C^{n-1}$ is the Drinfeld upper half space of the indicated dimension, C is the completion of the algebraic closure of $\mathbb{Q}_p$, and $\Pi_n := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \times \mathrm{GL}_n(\mathbb{Z}_p)$. Since we know the vanishing conjecture rationally, it is then enough to show that $H^1(\mathcal{H}_C^{n-1}; \mathcal{O}^+)$ is torsion-free. One way we have tried this is to show that $H^0(\mathcal{H}_C^{n-1}; \mathcal{O}^+/p) = \mathcal{O}_C/p$, which if true won't be purely formal since $H^0(\mathcal{H}_{\mathbb{Q}_p}; \mathcal{O}^+/p) \neq \mathbb{F}_p$. Another option that we are exploring is to use the an exponential sequence argument to try and reduce the computations of the p-torsion of $H^1(\mathcal{H}_C^{n-1}; \mathcal{O}^+)$ to computations of $H^1(\mathcal{H}_C^{n-1}; \mathcal{O}^{**})$ and various Tate twists, all of which are already known.

Iterated localizations of E-theory and higher dimensional Fargues–Fontaine.
Our problem group attempted to understand successive localizations of Morava E-theory. The approach was to try to set up similar machinery to Scholze-Weinstein by considering an analogue of the Fargues-Fontaine curve - however, this "curve" must live over rings like $k = \pi_0(L_{K(0)}L_{K(1)}E_2)$. This is a field and a topological space, but the multiplication on this field is not continuous, so k is not a Huber ring. Moreover, the presence of two "generators" means that the associated space must be a "Fargues-Fontaine surface", rather than a curve. During the week, we discussed what examples of "perfectoid" rings might look like in this context, and how to formalize it in terms of condensed mathematics.

A height 1 two towers isomorphism.

The known two tower correspondence connects the Lubin-Tate space to the Drinfel'd projective plane by building two infinite towers, and showing there is an isomorphism at infinite level as $(\mathrm{GL}_h(\mathbb{Q}_p) \times \mathbb{G}_h)$-equivariant perfectoid spaces. The Scholze-Weinstein perspective on the two-tower correspondence is to first build a pro-etale perfectoid space in terms of modifications. Then, their modification gives you a matrix A. They take the quotient corresponding to taking the kernel of A, and the quotient corresponding to taking the kernel of A transpose to get the Lubin-Tate and Drinfel'd spaces respectively. The two tower correspondence necessarily passes to generic fibers, giving us control of cohomology information only after inverting p.

Our group asks what can be done without inverting p. We start with the Lubin-Tate ring of height 2, i.e., $\pi_0(L_{K(2)}\mathbb{S})$., then mod out by p and invert u_1 to introduce a topologically nilpotent element, i.e., $\pi_0(L_{K(2)}L_{K(1)}\mathbb{S})$. We show this ring represents a moduli problem, p-divisible groups with an abstract trivialization, then build a tower replacing level structure by fixing the trivialization. We have shown for height 2 that this moduli problem at infinite level is equivalent to a moduli problem reformulated in terms of modifications, allowing us to directly apply Schloze-Weinstein's perspective to give us a modification theoretic description of all the creatures in sight, including the rather mysterious "Drinfel'd" side. We are now attempting to understand the moduli problem we get from this perspective on the "Drinfel'd" side; we hope it is a simpler function space with a simpler group action. We are further exploring the exciting possibility that our moduli problem carrying an action of $(\mathcal{O} \oplus \mathcal{O}(1)) \times (\mathcal{O}(1/2))$ sheds light on the equicharacteristic Jacquet-Langlands correspondence.

An algebraic proof of the exponential relation.

In their paper, Barthel, Schlank, Stapleton, and Weinstein make use of the fact that Morava E-theory admits symmetric powers, which are operations $\beta_m \colon E^0 \to E^0$ constructed using power operations and ambidexterity, both of which are deep structures possessed by E-theory. Drinfeld requested a construction of these symmetric powers (as integral operations) making use of tools in arithmetic geometry. There is a natural approach to answering this question. Ganter showed that there is an exponential relation

$$\sum_{m\geq 0} \beta_m t^m = \exp(\sum_{k\geq 0} \frac{T_{p^k}}{p^k} t^{p^k}),$$

where T_{p^k} is the p^kth Hecke operator. The goal of the project is to use this exponential relation to show that the resulting β_m's are integral.

This project was worked on for two of the afternoons of the conference. We expanded $\exp(\sum_{k\geq 0} \frac{T_{p^k}}{p^k} t^{p^k})$ in order to obtain congruences that needed to be satisfied by the Hecke operators. These congruences seemed to be related to some previous work of Stapleton's. Hesselholt suggested that Rezk's $\mathbb{T}$-monad and congruence criterion would be relevant to this problem and at this time, this seems like the most promising approach.