Cluster algebras and braid varieties

organized by
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Workshop Summary

Goals of the workshop

The workshop focused on the interplay between algebro-geometric and combinatorial properties of certain cluster varieties and the symplectic geometry of corresponding Legendrian links. The varieties in question are called *braid varieties of type A*, thanks to their definition in terms of words in the braid group Br_n, and can be defined in terms of configurations of flags. On the symplectic side, they appear as augmentation varieties of Legendrian contact differential graded algebras, as well as moduli spaces of certain microlocal sheaves.

The main goal was to build a bridge between two communities: researchers working on total positivity and cluster algebras, on one side, and those studying Legendrian links and exact Lagrangian cobordisms, on the other side. The workshop brought together specialists in both of these areas as well as several researchers working on adjacent topics. Both established and young researchers were present, which led to various discussions, insights, and many new collaborations. The dictionary between cluster combinatorics and geometry and properties of Legendrian links develops and continues to influence both of the areas, as will be illustrated by detailed explanation of topics of collaborations which grew from the working groups at the workshop.

Focus points of the working groups

During the week, we divided into several groups discussing various aspects of braid varieties. There were seven groups in total, although not every group met every day and there was a fair amount of interaction among the groups. In the following, we briefly describe the problems each group worked on, the progress during the workshop, as well as future directions.

Throughout, let β be a positive braid with Demazure product w_0 (the longest word in the symmetric group), and let $X(\beta)$ be the corresponding braid variety. Many of the threads below may extend to arbitrary type; for simplicity, we state everything in type A.

Peripheral locus of positroid varieties.
Let $X := \mathrm{Spec}(\mathcal{A})$ be a cluster variety, where $\mathcal{A}$ is a cluster algebra (with a specified cluster structure). Every cluster $\mathbf{x} \subseteq \mathcal{A}$ determines an open torus

$$\mathbb{T}_{\mathbf{x}} := \mathrm{Spec}(\mathcal{A}[\mathbf{x}^{-1}]) \subseteq X$$

known as a *cluster torus*. The *cluster manifold* is defined to be the union of all such tori:

$$M := \bigcup_{\mathbf{x} \text{ a cluster}} \mathbb{T}_{\mathbf{x}}.$$

The *peripheral locus* (or simply *periphery*) of X is defined to be the complement of the cluster manifold in the cluster variety:

$$P := X \setminus M.$$

The goal of this project is to compute the peripheral locus of positroid varieties, that are special cases of braid varieties. During the week at AIM we obtained that, if $\Pi^\circ_{2,n}$ is the maximal positroid stratum in the Grassmannian $\mathrm{Gr}(2,n)$, then

$$P(\Pi^\circ_{2,n}) = \begin{cases} \emptyset & n \text{ odd} \\ \mathbb{C}^{n-2} & n \text{ even.} \end{cases}$$

This result already has interesting consequences in symplectic geometry, for it implies that the Fukaya category for $\mathrm{Gr}(2,2n+1)$ is generated by finitely many Lagrangian tori, which then implies the validity of Kontsevich's homological mirror symmetry for $\mathrm{Gr}(2,2n+1)$.

We are continuing to work on this direction, in particular working now with the maximal positroid variety $\Pi^\circ_{3,n}$ in the Grassmannian $\mathrm{Gr}(3,n)$. We conjecture that $P(\Pi^\circ_{3,n})$ is empty whenever 3 does not divide n and, more generally, that $P(\Pi^\circ_{k,n}) \subseteq \mathrm{Gr}(k,n)$ is empty whenever n and k are coprime.

Since the workshop, we have been made aware that there is another group of researchers working on properties of the periphery of cluster varieties, which they call "deep points". It appears that the results obtained by the two groups are largely complementary.

Cluster compactifications of braid varieties.

Let us consider a positive braid *word* on n strands $\mathbf{i} \in \{1,\dots,n-1\}^\ell$. Associated to this word we have two varieties: the *braid variety* $X(\mathbf{i})$ (that depends only on the positive braid β defined by $\mathbf{i}$) and the *brick variety*, $\mathrm{brick}(\mathbf{i})$ [escobar2016brick], that was defined by A. Knutson in his talk, and depends on the braid word $\mathbf{i}$. We have a natural inclusion $X(\mathbf{i}) \subseteq \mathrm{brick}(\mathbf{i})$, and the latter variety is a compactification of the former. Given that $X(\mathbf{i})$ is a cluster variety, the following is the question that kickstarted the group:

Is $\mathrm{brick}(\mathbf{i})$ a cluster compactification of $X(\mathbf{i})$?

Where cluster compactifications were defined in L. Bossinger's talk. Throughout the week, we concluded that the answer to this question is *Yes* and, moreover, the potential for this compactification is given by:

$$W = z_1 + \cdots + z_\ell$$

where $z_1,\dots,z_\ell$ are natural coordinates on the braid variety $X(\mathbf{i})$. The rigorous proof of this uses the facts that (i) all z_i are cluster monomials and (ii) there exists a seed for which z_1 can be expressed as a product of cluster variables divided by a single frozen variable, which is moreover a source in the seed.

Questions to consider moving forward are:

(1) Does there exist a seed where W has a nice polytopal realization? The z_i's are cluster monomials corresponding to *different* seeds, in general. An idea, explained to us by D. Weng, is to use a different coordinate system for cluster tori given by "pinching sequences" that appears naturally from the symplectic topological perspective.

(2) The Jacobi ring of W should give the quantum cohomology of the mirror to brick($\mathbf{i}$), but this should already include the quantum parameters. How do we recover these?
(3) A. Knutson defined in [knutson-slides-stable] another compactification of open Richardson varieties using moduli spaces of stable maps. Does this also appear as a cluster compactification?

Totally nonnegative parts of braid varieties.

A cluster variety has a naturally defined totally positive part, which is the locus where all cluster coordinates are positive real numbers. The group discussed what exactly are the coordinates that are assigned to the edges of a weave, and what would it mean for them to be positive. In particular, it was realized that the coordinates are naturally assigned to *oriented* edges of the weave, so the notion of positivity depends on the orientation. Demazure weaves are naturally oriented "downwards", but the notion of orientation is unclear for other types of weaves.

In a parallel direction, brick varieties are naturally stratified by braid varieties, and the group discussed what the totally non-negative (TNN) part of brick varieties are. In particular, a rigorous definition of the TNN part of brick varieties was proposed, and we showed that the TNN part is compact. It seems reasonable to expect that the TNN part of a brick variety is a closed ball that admits a cell stratification with the closure relations as in the subword complex [knutson2004subword]. In particular, the TNN part should depend on the braid word chosen, and not just on the braid itself. Another version of the TNN part of the brick varieties was recently given by Bao and He in [bao2022total], and it would be interesting to compare these two notions of the TNN part of a brick variety.

Ruling stratification vs. cluster stratification.

With a Legendrian link Λ, one can associate, following Chekanov [chekanov2002differential], a differential graded algebra, also know as the *Legendrian contact DGA* of Λ. For links realizing (-1)-framed closures of positive braids in $\mathbb{R}^3$ with the standard contact structure, the moduli spaces of *augmentations* of such DGAs are affine algebraic varieties isomorphic to braid varieties associated with the corresponding braids. Henry and Rutherford [HR] showed that the augmentation variety of a Legendrian link Λ has a decomposition into $(\mathbb{C}^k) \times (\mathbb{C}^*)^l$ pieces. Each piece in this decomposition is indexed by a *graded normal ruling*.

On the other hand, each braid variety $X(\beta)$ have a (non-unique) decomposition into $(\mathbb{C}^k) \times (\mathbb{C}^*)^l$ pieces which comes from *weaves* [cggs]. The pieces in such a decomposition are indexed by certain "simplifying" weaves from β to Δ with k cups and l trivalent vertices. Each such decomposition is closely related to the cluster structure on these varieties, in particular, each piece in the decomposition is given by the intersection of the vanishing locus of certain cluster variables in a seed, with the non-vanishing locus of the remaining cluster variables in the same seed. A simplifying weave on a braid β encodes a set of augmentations of $\Lambda(\beta)$, the (-1)-framed closure of the braid β,via the moduli of local systems of the associated Lagrangian filling. In particular, if a simplifying weave has no cups then the set of augmentations it defined forms a cluster torus. Note that there are many stratifications of $X(\beta)$ given by simplifying weaves.

During the week at AIM, we defined a possible bijection between the indexing sets of these decompositions. That is, we chose a particular weave stratification, with pieces indexed by "left-inductive" simplifying weaves. We then defined a map between the rulings for Legendrians associated to the (-1)-closure of β, and the left-inductive simplifying weaves for β. We expect that this bijection extends to a geometric identification of the two stratifications, up to the identification of $X(\beta)$ with the appropriate augmentation variety, and are continuing to work in this direction.

Combinatorics of weaves and 3D plabic graphs.
There are two recent independent constructions of a cluster structure on $X(\beta)$ [cgglss,glsbs,glsb]. In the first construction, seeds are encoded by weaves; those encoded by *Demazure* weaves are best understood. In the second construction, seeds are encoded by 3D plabic graphs. At present, the relationship between these two constructions is unknown. We began by comparing the combinatorics, with the expectation that this will lead to an understanding of the geometry. Among the topics discussed were:

- Is there a combinatorial procedure to produce a (inductive, Demazure) weave from a 3D plabic graph, a la the procedure for plabic fences in [CL]?
- Relatedly, forthcoming work of Casals–Le–Sherman-Bennett–Weng will give a construction of a Demazure weave from a reduced plabic graph by iterating a combinatorial procedure known as *T-duality*, which arises in the study of the $m = 2$ amplituhedron [PSBW] and zonotopal tilings [Gal$_t$$ilings].CanT - dualitybeextendedtonon - reducedplabicgraphs?InsuchawaythatiteratingT{-}dualityproducesthecorrectweave?Inthesetting$
- Can arbitrary Demazure weaves be encoded in something similar to a zonotopal tiling?
- We also compared the combinatorics of Lusztig cycles in (inductive) Demazure weaves and "soap films" in 3D plabic graphs. We defined a possible bijection between the two, and justified why braid moves in some horizontal slice of the weave should not affect the soap film at a certain slice of the 3D plabic graph. We expect to make a precise dictionary between the two combinatorial objects.

Lagrangian cobordism of positroid links.

Casals–Gorsky–Gorsky–Simental realized open positroid varieties of the complex Grassmannian as augmentation varieties of Legendrians called positroid links [cggs2]. In a project which started during the workshop and led to the preprint [asplund2023lagrangian], it was proved that the partial order on positroid varieties induced by Zariski closure also has a symplectic interpretation, given by exact Lagrangian cobordisms of positroid links.

Cluster modular group of braid varieties.
The group on cluster automorphisms of braid varieties investigated cluster automorphisms for positroid varieties. The proposed strategy is to use the covering relation that the Lagrangian cobordism group worked out to try to see when a particular cluster automorphism on the Grassmannian might or might not extend to certain positroid strata, or when a new automorphism might appear that had not existed for higher strata.

Outcome

The workshop contributed to the interaction between the cluster algebra community and symplectic geometers and led to better understanding of concepts and problems on one side by the researchers from the other area, and vice versa. While there are many open problems and not all connections have been made clear yet, we believe that the workshop was a success in connecting the communities and research areas.

The open problem session and the working group topics helped us to form the problem list maintained by **Angela Wu** and available at the website of AIM [problems].

All the groups which were meeting during the workshop continue to work on the projects mentioned in the above section. The first direct outcome of the collaborations started at the workshop is a joint paper of the group investigating Lagrangian cobordisms of positroid links:

- **Johan Asplund, Youngjin Bae, Orsola Capovilla-Searle, Marco Castronovo, Caitlin Leverson, and Angela Wu** wrote [asplund2023lagrangian].