Discrete and combinatorial homotopy theory

organized by
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Workshop Summary

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AIM Workhshop Report Discrete and Combinatorial Homotopy

WORKSHOP SUMMARY

The aim of this workshop was to begin to consolidate the numerous groups studying different models of discrete algebraic topology, and in particular homotopy theory in graph theory and combinatorics. To the best of our knowledge, this was the first workshop specifically devoted to this area, and bringing together these different groups for talks and discussions, most of whom had been previously working largely in isolation from one another, proved to be extremely fruitful.

Each morning from Monday to Wednesday, there were two talks, on Thursday monning there was a single talk followed by a group discussion, and on Friday morning, there were two regular talks followed by an additional, closing talk. The goal of the presentations from Monday to Wednesday was to give an overview of the different models of homotopy which had been studied in the area, and in particular for graphs, and the talks on Thursday and Friday were slightly more specialized.

The speakers were:

- Monday - Nikola Milicevic and Conrad Plaut
- Tuesday - Daniel Carranza and Laura Scull
- Wednesday - Nicholas Scoville and Anton Dochtermann
- Thursday - Federico Vigolo (followed by a group discussion)
- Friday - Ling Zhou, Chris Kapulkin, and Curtis Greene/Eric Babson, the latter two of whom gave a joint talk

On Monday afternoon, there was a problem session moderated by Antonio Rieser, which generated a list of around 14 open problems and research topics.

From Tuesday to Friday during the afternoons after the talks, the participants in the workshop divided into groups and worked on the following 5 problems.

Computing Homotopy Groups of the Digital Sphere

This group worked to define and compute several homotopy groups of a digital sphere. For some digital image X (a reflexive undirected graph on some lattice), the homotopy group $\pi_n(X, x_0)$ was defined to be the homotopy classes of maps $\alpha : I \to X$, where $I \subset \mathbb{Z}^n$ is a rectangular subset of the lattice and α maps the boundary ∂I to the point x_0.

Let $X \subset \mathbb{Z}^3$ be the 6-point set consisting of the basis vectors $\pm e_i$. When considered with diagonal adjacencies, X resembles the octahedron, which is a typical stand-in for a "digital sphere". The digital homotopy group $\pi_2(X, x_0) \cong \mathbb{Z}$ was computed using an elementary argument defining a combinatorial analogue of the degree for the map $\alpha : I \to X$. The constructions and arguments used are reminiscent of Sperner's lemma.

Higher Homotopy Groups for ×-Homotopy

The second group worked on defining the notion of (higher) homotopy groups in the ×-homotopy theory of graphs with (optional) loops. Previously, Chih and Scull had defined the fundamental group(oid) of a graph, so this group's goal was to generalize this earlier work.

They proceeded to define the loop graph of a graph via the path space. The definition takes as input a graph with a map to $(Z/2)^n$ together with a chosen vertex and chosen edge, and produces a new graph with a map to $(Z/2)^(n+1)$ along with a canonically chosen looped vertex, which, in turn, produces the distinguished edge. By iterating this construction, they obtained higher loop spaces, and they then defined the higher homotopy groups as connected components of the corresponding higher loop space.

Our definition makes the n-th homotopy group admit natural $(Z/2)^n$ grading. One direction that appears to be worth exploring further is whether this grading could be used to prove the non-existence of certain maps, which could have potential applications to colorability.

Much of the work was spent refining the definition, and the group has a number of ambitious goals ahead, including: (1) Proving that higher homotopy groups are abelian, (2) Constructing the Hurewicz homomorphisms and proving the Hurewicz theorem, and (3) Relating their construction to the one of Dochtermann in case of fully looped graphs.

Homology Theories for General Graphs

The problem explored by this group was to search for a good construction for homology of graphs with (optional) loops, and they explored several possibilities. Noticing that two of their possible constructions gave isomorphic results, both definitions arising from the homology of certain simplicial complexes associated to the graph. They then conjectured that the two complexes are in fact homotopy equivalent and sketched the proof of this fact. It was then verified that this fact was well-known to experts, and, in particular, had appeared in a paper by Eric Babson.

A-Theory Weak Equivalence of Suspensions of Different Length

This problem asks whether a natural map $f : \Sigma_{n+1}(G) \to \Sigma_n(G)$ induces an isomorphism from $A_d(\Sigma_{n+1}(G), v_0)$ to $A_d(\Sigma_n(G), v_0)$, for $d \geq 0, n \geq 3$. Here G denotes a graph, $\Sigma_n(G)$ denotes the quotient $I_n \Box G/\sim$, where I_n is a path with vertices $0, 1, \ldots, n$, $\Box$ denotes box product, and $\sim$ identifies the top and bottom levels to single points.

The group explored the problem using a diagrammatic template suggested by Kris Kapulkin (the proposer), initially considering the special case where $G = Z_5$, the pentagon graph, and $n = 3$. Specifically, given a graph map $\sigma : I^d_k \to \Sigma_3(G)$, where I^d_k is a d-dimensional grid of length k, we must show that, up to homotopy, any lifting of σ to $\Sigma_4(G)$ on the boundary of I^d_k can (after expanding I^d_k to a larger grid), be extended to the interior.

$$\partial I^d_{3k+2}[d, hook][r]\partial I^d_k[r, "\tilde{\sigma}"][d, hook]\sum_4(G)[d, “f”]$$

$$\mathrm{I}^{d}_{3k+2}[r, “j”'][rru, dashed, “g_\sigma”]I^{d}_{k}[r, ”\sigma”'] \sum_3(G).$$

They found algorithms for constructing such an extension in dimensions $d = 1, 2$, and 3, proving that the induced map f^* is an isomorphism in dimensions 1 and 2, and that it is surjective in dimension 3. More generally, the algorithm works if $d \leq n$, but may fail otherwise. However, they found no evidence that the result is false, and no obstacle to this overall approach.

Homotopy at scale and closure spaces

The objective of this group was to understand the similarities and differences between multiple approaches to coarse homotopy groups on metric spaces. It is known that there is not an algorithm to compute any of them as it would solve the word problem in groups, and it is easy to verify that all of them are close in some sense (interleaving distance).

The setting is the following: One has a pointed metric space $(Y, *)$, an integer k, and a scale parameter $\varepsilon > 0$. One then wants to compute the k-th homotopy group of $(Y, *)$ while disregarding whatever happens at scale $< \varepsilon$. With the closure structure $c_{\varepsilon,-} : \mathcal{P}(Y) \to \mathcal{P}(Y)$ given by $c_{\varepsilon,-}(A) := \{y \in Y | d(y, A) < \varepsilon\}$, one could study the following:

(1) A-theory groups $A_k((Y, c_{\varepsilon,-}), *)$.
(2) $\times$-homotopy groups $\pi^{\times}_k((Y, c_{\varepsilon,-}), *)$.
(3) Vietoris–Rips homotopy groups $\pi_k(VR_{<\varepsilon}(Y), *)$.
(4) closure space homotopy groups $\pi_k((Y, c_{\varepsilon,-}), *)$.
(5) semi-uniform space homotopy groups $\pi_k((X, \mathcal{F}_\epsilon), *)$.
(6) ε-continuity homotopy groups $\pi^{\varepsilon}_k(Y, *)$.

Approach 6 is isomorphic to certain special cases of approach 5, and we conjecture that 3 is isomorphic to 5, but a proof seems very hard. Approaches 1 and 2 appear to be able to be reduced via suitable loop spaces:

$$A_k(X, *) = A_{k-1}(\Omega^A(X), *), \;\; \pi^{\times}_k(X, *) = \pi^{\times}_{k-1}(\Omega^{\times}(X), *).$$

There was an attempt to obtain such reduction for approach 4. The main issue is that the set of continuous maps in the category of closure spaces from S^1 to $(Y, c_{\varepsilon,-})$ is not yet known to be a closure space, as it is only a priori a pseudo-topological space. What one would like is something of the form

$$\pi_k((Y, c_{\varepsilon,-}), *) = \pi_{k-1}(Hom^*_{Cl}(\mathbb{S}^1, Y, c_{\varepsilon,-}), *).$$

The group has conjectured that when Y is a nice space, such as a Riemannian manifold, one could substitute approach 6 by $\pi^{\varepsilon}_{k-1}(\Omega(Y), *)$. This is inspired by the classical approach when one shows that $\Omega(Y)$ is homotopy equivalent to a Riemannian manifold where standard analysis tools such as Morse theory are available.