Computations in stable homotopy theory

organized by
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Workshop Summary

Background

Our workshop, *Computations in Stable Homotopy Theory*, focused on a range of computer-assisted approaches to problems in homotopy theory. Participants were interested in different kinds of computational questions, and many discussions centered on how computational tools can support these projects. The topics raised included both practical aspects, such as programming, algorithms, and computational resources like GPUs, as well as more theoretical considerations related to these computations.

Structure of the workshop

The workshop gathered participants across all career stages—from PhD students and postdocs to senior researchers. The organizers also aimed to balance participants working on different computational problems in homotopy theory, reflecting both the practical and theoretical sides of computer-assisted methods.

The program ran from 9 am to 5 pm, Monday through Friday. The morning talks followed this schedule:

- 9 am Monday: Zhouli Xu, *Differentials in the Adams spectral sequence*
- 11 am Monday: Weinan Lin, *Machine Computations of Adams Spectral Sequences and Extensions*
- 9 am Tuesday: Zhouli Xu, *Proof for the existence of* θ_6
- 11 am Tuesday: Weinan Lin, *Generalized Leibniz rule and generalized Mahowald trick*
- 9 am Wednesday: William Balderrama, *Unstable synthetic homotopy theory*
- 11 am Wednesday: Danny Xiaolin Shi, *Periodicity and finite complexity in higher real* K*-theories*
- 9 am Thursday: Dan Isaksen, *Machine computation in unstable homotopy*
- 11 am Thursday: Ben Antieau, *Computability of* ∞*-categories*
- 9 am Friday: Bert Guillou, *The homotopy of Klein-four normed spectra*
- 11 am Friday: Sarah Petersen, *Equivariant and motivic splittings of truncated Brown–Peterson spectra*

In the afternoons, participants were divided into groups to work on problems chosen collectively.

Project group summaries

Improving performance of Milnor Multiplication.
Our group investigated approaches for accelerating multiplication in the Steenrod algebra, which is the bottleneck for Adams spectral sequence computations. We considered the Milnor basis, and discussed approaches to an algorithm for Steenrod multiplication that

takes advantage of the structure of Milnor matrices. The main idea of the algorithm is to compute the Milnor matrices "one bit at a time" using a backtracking approach. We also discussed parallelism approaches via work stealing queues. We also discussed how to implement such an algorithm on the GPU. We intend to implement such algorithms and test them with state-of-the-art Adams spectral sequence software.

Computer methods for chromatic spectral sequences.
The name "chromatic homotopy theory" was coined by Miller–Ravenel–Wilson when they invented the chromatic spectral sequence (CSS) to study the E_2-page of the Adams—Novikov spectral sequence. The input of the CSS is computed by a sequence of Bockstein spectral sequences, starting from the Ext groups of $v_n^{-1}BP/(p, v_1, \cdots, v_{n-1})$. The resulting group $\mathrm{Ext}^0(v_n^{-1}BP/(p^\infty, \cdots, v_{n-1}^\infty))$ is known as the group of Greek letter elements of height n.

During the workshop, we revisited their program to compute the α and β families. I explained my SageMath code to recover the Miller–Ravenel–Wilson computation of $\mathrm{Ext}^0(v_n^{-1}BP/(p, \cdots, v_{n-2}, v_{n-1}^\infty))$ and possible rooms to improve the algorithm. Sihao Ma explained the Shimomura program to compute homotopy groups of the $K(2)$-local spheres at primes $p \geq 5$ using Bockstein spectral sequences. We proposed two follow-up projects to work on using machine-assisted methods:

- Compute the mod-p γ-family elements following the Miller—Ravenel—Wilson program.
- Compute the homotopy groups of the $K(2)$-local half spheres $E_2^{h\mathbb{G}_2^1}$ following the Shimomura program. It is noted that even their rational homotopy groups are still an open problem.

The long-term goal is to write a computer program to run general Bockstein spectral sequences in homotopy theory.

Differential propagation in the slice spectral sequence.
The goal of the project is to compute the C_{2^k}-slice spectral sequence for $EO_n(C_{2^m})$ for various values of k, m, and n. These spectral sequences demonstrate unexpected rigidity in the following sense. The spectral sequences are known to have certain properties, such as a vanishing region; a Leibniz rule for products; specific periodicity properties; and a transchromatic isomorphism that relates the spectral sequences for different values of k, m, and n. In some small cases, one can reverse-engineer a unique pattern of differentials that is consistent with these properties.

However, the details of the arguments are complex. Recently, machines have been shown to be effective at exhaustively deducing information about differentials in spectral sequences from properties. The goal of the project is to apply machine deduction to study more difficult cases of the C_{2^k}-slice spectral sequence for $EO_n(C_{2^m})$.

The ultimate goal of the project is the computation of the coefficients for the higher real K-theories, which generalize the theories $\mathrm{H}\mathbb{Q}$, ko, and tmf that play central roles in computation. The difficulty at higher chromatic heights is the intricacy of the computations involved.

Note that the detection spectrum for Hill-Hopkins-Ravenel is the case $n = 4$ and $m = 3$. It is at least plausible that machines could completely compute this case. This work would also yield exact values of the slice differentials on the elements θ_j for $j \geq 7$.

Secondary Steenrod algebra.
There has been a recent resurgence in interest in the secondary Steenrod algebra, which can be used to algorithmically compute the E_3 page of the Adams spectral sequence in a range, much as (Ext over) the Steenrod algebra computes the E_2 page. This algorithm has been implemented by Dexter Chua and separately by Weinan Lin to compute the E_3 page of the Adams spectral sequence for the sphere, and select other finite complexes, at the prime 2. While the theory of secondary operations is a classical topic of study, the theory of the action of the secondary Steenrod algebra as a whole on secondary cohomology is much less developed.

After reviewing the literature about the secondary Steenrod algebra $\mathcal{B}$ and working out some small examples, we focused on the problem of how to compute secondary cohomology; so far, there are very few techniques aside from showing via obstruction theory that the $\mathcal{B}$-module structure is in some sense trivial. One idea was to develop some of the standard machinery for ordinary cohomology (Mayer-Vietoris, cellular homology, long exact sequences) in the setting of secondary cohomology. This is more or less straightforward when one uses a too-big-to-be-useful definition of secondary cohomology, but the real challenge is in determining how these constructions interact with strictification, which produces computable models. Another approach was to use known information about d_2 differentials of certain finite complexes to determine part of the $\mathcal{B}$-module structure. We believe that even partial improvements in computational techniques would lead to real gains in the systematic machine computation of Adams spectral sequences.

Anderson dual of $L_n BP^{((G))}$.
$\langle m \rangle$ Hill–Meier showed that the topological modular forms with level structure $tmf_1(3)$ is a form of $BP_{\mathbb{R}}\langle 2 \rangle$ and computed $Tmf_1(3)$ is self C_2-equivariant Anderson dual up to an $RO(C_2)$-shift of $5+2\rho$. This result can be interpreted as a computation of the C_2-equivariant Anderson dual of $L_2 BP_{\mathbb{R}}\langle 2 \rangle$. Beaudry–Hill–Shi–Zeng showed that the quotient normed Brown–Peterson spectrum $BP^{((C_{2^n}))}\langle m \rangle$ is a C_{2^n}-spectrum of maximal height $h = 2^{n-1}m$. The goal of this project is to compute the C_2-equivariant Anderson dual of $L_h BP^{((C_{2^n}))}\langle m \rangle$.

During the workshop, we tried to write the $E(h)$-local of $BP^{((C_{2^n}))}\langle m \rangle$ as a limit of a diagram of spectra like $BP^{((C_{2^n}))}\langle m \rangle$ with certain elements inverted. Once we have this, we plan to take the slice filtration of each spectrum in the limit diagram to make it a diagram of filtered spectra. This induces a filtration on the limit $L_h BP^{((C_{2^n}))}\langle m \rangle$. Although this is not the slice filtration in general, we outlined a proof sketch showing that it commutes with the equivariant Anderson dual. This filtration is useful because the associated graded pieces of the slice filtration of $BP^{((C_{2^n}))}\langle m \rangle$ and related spectra are suspensions (including $RO(G)$-suspensions and smashing with finite G-sets) of $H\underline{\mathbb{Z}}$(of various subgroups), and the equivariant Anderson dual of $H\underline{\mathbb{Z}}$ is known to be $\Sigma^{2-\lambda_0} H\underline{\mathbb{Z}}$ for cyclic groups of order 2^k (λ_0 is 2σ when $k = 1$).

Computability of ∞-categories.
During the week, we discussed how to obtain a computable model for certain ∞-categories of generalized Eilenberg-MacLane spaces and n-types in the case of small n. Our approach centered around leveraging the adjunction between simplicial sets and chain complexes,

which offered a more concise description of objects on the chain complex side at the cost of a more intricate description of morphisms and composition via the adjunction.

Various ∞-categories admit the structure of comodules over a comonad on the differential graded nerve of a category of chain complexes. However, it is not clear from a theoretical standpoint when the Kleisli category of a nonadditive comonad on a differential graded category admits a description in terms of the differential graded nerve. In the cases where such a description does exist, our discussions suggest that our hybrid approach—combining concise chain-level models for objects with adjunction-based descriptions of morphisms—or, alternatively, viewing objects as left modules over a (possibly complicated) comonad—is likely to be more effective computationally than representing objects via iterated bar constructions. We are optimistic that advances in our understanding of ∞-category theory as well as in computing (both hardware and software) will make it feasible to carry out explicit machine computations within ∞-categories.

Hurewicz image of H_{C_2}.
$\underline{\mathbb{F}}_2$ Our group worked on determining the C_2-equivariant $RO(G)$-graded Hurewicz image of $H\underline{\mathbb{F}}_2$, a problem that, somewhat surprisingly, had not been settled prior to the workshop. Beginning with the 0-line of the C_2-equivariant genuine Adams spectral sequence as computed by Guillou–Isaksen, we applied Ma's theorem, which relates the genuine and Borel Adams spectral sequences—both at the level of their E_2-pages and in the pattern of differentials. This allowed us to translate the question into understanding the 0-line of the classical Adams spectral sequence for stunted projective spaces. From there, Adams' classical solution to the vector fields problem on spheres provides the final answer.

We were delighted to obtain a complete description of the Hurewicz image in this setting. At the same time, we observed that the 0-line supports many interesting and nontrivial differentials that remain to be understood.

Tools for Adams spectral sequences for non-flat ground rings.
Our group discussed various approaches to computing Adams spectral sequences with non-flat ground rings. We particularly focused on understanding the new approach of using quivers, developed by Burklund-Pstragowski. Their paper develops the theory, and works out the example of an $H\mathbb{Z}$-based Adams spectral sequence. A natural next example to try is connective complex k-theory, ku, both because Gonzalez's ku-based Adams spectral sequence computations using older techniques have been very fruitful, and because ku is a height one analogue of $H\mathbb{Z}$.

The first step in this analysis for a given ground ring R is to understand its quiver, which encodes all of the modules produced by tensoring R with itself repeatedly, and the relations among those modules. This is significantly more complicated for ku than for $H\mathbb{Z}$. During the AIM workshop, we worked out a conjecture for the structure of the quiver for ku. Next steps would be to confirm that this is indeed the quiver for ku, and then use this quiver to compute with the ku-Adams spectral sequence. After that, it would be interesting to extend these techniques to real connective k-theory, ko, as its quiver is a slightly more complicated version than that of ku and carries some additional information.

Indeterminacy of 4-fold Toda brackets.
The detailed analysis of finite cell complexes requires the study of obstructions that are elements of Toda brackets. We study Massey products in the homology of a differential

graded algebra as an approximation to the behavior of Toda brackets in stable homotopy theory.

Threefold Massey products have a satisfying theory, but even fourfold Massey products exhibit some subtle behavior that has not been fully described. In particular, a fourfold Massey product is not necessarily an affine subspace of a homology group. In other words, the indeterminacy is not necessarily a linear subspace.

The goal of the project is to inspect the defining systems for fourfold Massey products and to describe the indeterminacy in concrete terms. The main potential application is to the analysis of 5-cell complexes via fourfold Toda brackets.

Our group conducted a detailed study of the indeterminacy of 4-fold Massey products, articulated through several equivalent formulations, and we compared these descriptions with known cases arising in the Adams spectral sequence. The preliminary discussions at AIM led to some tangible new results, although more needs to be done to solve the problem.