Smooth concordance classes of topologically slice knots

organized by
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Workshop Summary

Knot and link concordance has been studied since the early 1960s because of its strong relationship to the surgery-theoretic scheme for classifying 4-dimensional manifolds. Freedman's celebrated disk embedding theorem combined with significant advances in smooth 4-manifold theory led us to discover highly non-trivial structure within the set of smooth concordance classes of topologically slice knots. Nevertheless, we are far from having a complete understanding of this set. The primary goal of the workshop was to advance and deepen our understanding of the relationship between smooth and topological concordance by facilitating the free exchange of ideas among researchers with varied expertise.

Morning talks

The workshop participants were all interested in studying knot concordance, but come from various backgrounds. Broadly speaking, roughly half of the participants were well-versed in techniques from the topological category, and the other half in techniques from the smooth category. The goal of the morning talks was to introduce each half of the audience to questions, ideas, and techniques from the other.

Filtrations of the knot concordance group.
Kent Orr and Jae Choon Cha gave introductory talks on filtrations of the knot concordance group. Orr presented the definition of the solvable filtration introduced by Cochran-Orr-Teichner [timCOT], presenting the motivation coming from Freedman's disk embedding theorem. Cha gave a survey of known results about the set of smooth concordance classes of topoligically slice knots, denoted by $\mathcal{T}$. This led to a natural question: how can we systematically understand $\mathcal{T}$? Cha then defined the bipolar filtration introduced by Cochran-Harvey-Horn [timCHH2] as a possible framework. Lastly, he listed some recent non-triviality results for the bipolar filtration, including his own recent work on the primary decomposition of the bipolar filtration.

Construction of locally flat disks for Alexander polynomial one knots.
Mark Powell sketched a proof of the well-known result of Freedman's [timFreed1] which says that every knot with trivial Alexander polynomial bounds a locally flat disk $\Delta \subset B^4$ where $\pi_1(B^4 \smallsetminus \Delta) \cong \mathbb{Z}$. In particular, he gave the proof by Garoufalidis-Teichner [Garoufalidis-Teichner:2004-1] which is inspired by Levine's method of slicing high-dimensional algebraically slice knots.

Slicing obstructions in the topological category.
Allison Miller and Christopher Davis gave talks on methods of obstructing the topological

sliceness of knots. Miller presented a method from the paper of Casson and Gordon [Casson-Gordon:1986-1] which first showed that there are algebraically slice knots which are not topologically slice. She also explains the difficulty of the computations of this obstruction and ways to overcome it (e.g. twisted Alexander polynomials [Kirk-Livingston:1999-1]). Davis presented a way to use L^2 ρ-invariants to show some knots are not topologically slice following the paper of Cochran-Orr-Teichner [timCOT]. This method was the the first to find examples of algebraically slice knots which are not topologically slice even when the obstructions from [Casson-Gordon:1986-1] all vanish. He explained how to find a bound on the higher order ρ-invariants of a knot using the ordinary signature of a different knot which is related to the original one.

Slicing obstructions in the smooth category.
Danny Ruberman, Jennifer Hom, András Stipsicz, and Lisa Piccirillo gave talks on methods of showing some knots are not smoothly slice. Ruberman explained a few obstructions coming from restrictions on the intersection form of smooth closed 4-manifolds. For instance, he talked about the first example of a topologically slice knot that is not smoothly slice which was discovered independently by Casson and Akbulut using the fact that the intersection form of a closed smooth definite manifold is diagonalizable [Donaldson:1987-1]. Further, he made a remark that using Furuta's 10/8-theorem [Furuta:2001-1] one can also get slicing obstructions (e.g. [Donald-Vafaee:2016-1]). He also talked about obstructions coming from Chern-Simons theory.

Hom gave a survey talk on how knot Floer homology introduced independently by Ozsváth-Szabó [Ozsvath-Szabo:2004-1] and Rasmussen [Rasmussen:2003-1] can provide slicing obstructions. She also made a remark that Heegaard Floer homology of a branched cover of a knot is not completely determined by the knot Floer homology of the knot, therefore in general homology cobordism invariants coming from Heegaard Floer homology (e.g. $d, \overline{d}, \underline{d}$, and HF_{conn}) give slicing obstructions.

Stipsicz talked about how Heegaard Floer homology of branched covers of knots can give even more slicing obstructions by considering the lift of the knots and an involution on the chain complex. He mainly focused on his recent work with Alfieri-Celoria and Alfieri-Kang.

Piccirillo presented her new method of obstructing a knot from being smoothly slice. This technique was used by her to show that the Conway knot is not smoothly slice; prior to her work, the Conway knot was the only knot under 13 crossings whose sliceness was unknown. The key lemma is the fact that a knot K is smoothly slice if and only if the 0-framed surgery trace of K smoothly embeds in S^4. Given a knot K, she presented a method to find a knot J so that the trace of K and the trace of J are diffeomorphic. Hence in order to show that K is not smoothly slice it would be enough to show J is not smoothly slice. She makes a remark that this can be done by using invariants which are not trace invariants (e.g. Rasmussen's s-invariant [Rasmussen:2010-1]).

Metrics on the knot concordance group.
Shelly Harvey gave a talk on metrics of the knot concordance group. The way of considering the knot concordance group as a metric space was introduced by Cochran and Harvey [Cochran-Harvey:2018-1] as a approach to understand the knot concordance group in a more systematic way. Different types of metrics were further studied by Cochran-Harvey-Powell

[CHP15] using the notion of gropes. She presented some known results and open problems in this direction.

Afternoon sessions

Monday afternoon consisted of a problem session moderated by Matthew Hedden. Allison Miller was the scribe. The questions included broad open-ended questions as well as more specific ones.

Starting on Tuesday, afternoons were spent working on specific problems in groups. For these sessions, the organisers selected twelve problems. We used the standard AIM format for dividing into smaller groups. Starting on Wednesday, we began the afternoon sessions by having each group from the previous day give a short report on their progress. We next summarize motivations and progress on a few problems from the list:

Which satellite operators are trivial?.
Any knot P in a solid torus defines a natural map $P\colon \mathcal{C} \to \mathcal{C}$ from the concordance group to itself; we call this map a *satellite operator*. A satellite operator $U\colon \mathcal{C} \to \mathcal{C}$ is called the *identity satellite operator* if the knot U in a solid torus is isotopic to the core of the solid torus. A satellite operator P is called a *trivial satellite operator* if P and U are the same as maps on $\mathcal{C}$.

It was proven in [Cochran-Franklin-Hedden-Horn:2013-1] that the 0-surgery on K is homology cobordant relative to its meridian to that on $P(K)$ if P is unknotted when the solid torus is embedded in S^3 in the standard way and P generates the first homology of the solid torus. This makes it difficult to distinguish $P(K)$ and K since they share many classical invariants. Nevertheless it was shown in [Cochran-Franklin-Hedden-Horn:2013-1] that there exists a satellite operator, namely the Mazur sattelite operator, which is not trivial on the smooth concordance group. Unfortunately, it is still not known if such examples in the topological category exist. One could ask very concretely whether the Mazur satellite operator is trivial on the topological concordance group or be more ambitious and ask if there is a general statement that would lead us to find many satellite operators that are trivial on the topological concordance group. One group changed the perspective slightly and considered the dual curve of P and asked whether if the dual curve of P is homotopic to the dual curve of the identity satellite operator U, then P is trivial. This strategy resembles the work of Cochrna-Friedl-Teichner [timCFT] where they constructed many interesting toplogically slice links by an operation called *multi-infection*.

Which knots bound disks in homology balls but not in the four-ball?.
Most slicing obstructions, including both classical and modern obstructions, vanish if a knot bounds a disk in a homology ball. It is a natural question to ask if there is a knot that bounds a disk in a homology ball (or even in a homotopy ball) but not in the four-ball. Little is known, in either category. A few suggestions were made for constructing examples. One possibility is to follow the proof of Fintushel and Stern [Fintusehl-Stern:1984-1] where they showed that the figure-eight knot bounds a disk in a rational homology ball even when it does not bound a disk in the four-ball. Another approach is to relate this problem to that of finding a knot in a homology sphere that does not bound a piecewise linear disk in one homology ball but bounds a smoothly embedded disk in a different homology ball. This perspective is closely related to Akbulut's result [Akbulut:1991-1] where he solved Zeeman's

conjecture. In order to obstruct such knots bounding a disk in the four-ball, Rasmussen's s-invariant [Rasmussen:2010-1] or the technique of Piccirillo [Piccirillo:2018-1] may be used.

Which satellite operators are homomorphisms?.
As we have observed above, given a knot P in a solid torus we get a satellite operator $P\colon \mathcal{C} \to \mathcal{C}$. Since $\mathcal{C}$ is a group, it is natural to ask when a satellite operator induces a group homomorphism on $\mathcal{C}$. It is conjectured by Hedden that P is a homomorphism if and only if P is either the zero map, the identity, or the involution coming from orientation reversal. Using the Levine-Tristram signature it can be easily deduced that if P is a homomorphism, then it either generates the first homology of the solid torus or is null-homologous. On a related note, it has been shown by Hedden and Pinzón-Caicedo that there are a lot of satellite operators with a very large image.

Are $(4_1)_{2,1.}$
and $Wh^+(T_{2,-3})$ *smoothly slice?* Let $(4_1)_{2,1}$ denote the $(2,1)$-cable of the figure eight knot and $Wh^+(T_{2,-3})$ the positive Whitehead double of the left handed trefoil knot. These are two of the simplest examples of knots where we do not know if they are slice or not. There are more reasons why these two knots are interesting. It was proved by Miyazaki [Miyazaki:1994-1] that $(4_1)_{2,1}$ is not homotopy-ribbon using the Casson-Gordon homotopy ribbon obstruction for fibered knots. In view of the slice-ribbon conjecture, it is natural to conjecture that $(4_1)_{2,1}$ is not slice. Also, $(4_1)_{2,1}$ is a rationally slice knot with complexity one and it is not known if there exists a complexity one rationally slice knot which is not slice. The Whitehead double operator is one of the simplest satellite operators and it is conjectured that the Whitehead double of a knot K is smoothly slice if and only if K is smoothly slice. Using various gauge-theoretic techniques it has been shown that Whitehead double of positive knots, as well as of knots satisfying other, weaker notions of positivity, are not slice. The sliceness of positive Whitehead doubles of negative knots, such as the left-handed trefoil, is unknown. One group enumerated the obstructions that vanish for these two knots and obstructions yet not known to vanish.

Is ribbon concordance a partial order?.
One of the most well-known and interesting conjectures in knot concordance is the slice-ribbon conjecture. A knot is said to be *ribbon* if it bounds a immersed disk in the three-sphere with only ribbon singularities. By definition, every ribbon knot is smoothly slice and the slice-ribbon conjecture asserts that the converse is true. A related notion is *ribbon concordance.* It is conjectured by Gordon [Gordon:1981-1] that ribbon concordance is a partial order on the set of knots. There has been some recent interest in this question, including the result of Zemke [Zemke:2019-1] that if K is ribbon concordant to J and J is ribbon concordant to K, then K and J have the same Seifert genus. A group spent some time trying to understand [Gordon:1981-1]. They learned that ribbon concordance is a partial order for a large family of knots, namely those where the commutator subgroup of the fundamental group of the knot complement is transfinitely nilpotent. They determined that to extend the group theoretic techniques used by Gordon they would perhaps need to use other group series such as Harveys torsion-free derived series. They also discussed an entirely different approach using sutured manifolds.

Is every knot in a homology sphere topologically concordant to a knot in the three-sphere?.
Every knot in the three-sphere bounds a piecewise linear disk in the four-ball. Then a natural question is whether this is still true for knots in homology spheres which bound

contractible manifolds. It was conjectured by Zeeman [Zeeman:1964-1] that the answer to this question is negative; Akbulut [Akbulut:1991-1] confirmed his conjecture. Further, A. Levine [Levine:2016-1] proved that in fact there exists a knot in a homology sphere that does not bound any piecewise linear disk in any contractible manifold. Note that a knot K bounds a piecewise linear disk if and only if K is smoothly concordant to a knot in the three-sphere. Therefore a natural analogue of this question in the topological category is whether every knot in a homology sphere is topologically concordant to a knot in the three-sphere. A group followed the proof of the theorem of Freedman [timFQ] where he proved that every homology sphere bounds a contractible manifold and investigated if this proof can be generalized to answer the question.

Does either homology cobordism classes of Dehn surgeries or branched covers of a knot determine its concordance class?.
If two knots K and J are concordant, then the p/q-surgeries on K and J, denoted by $S^3_{p/q}(K)$ and $S^3_{p/q}(J)$ respectively, are homology cobordant. Additionally, if two knots K and J are concordant and p is prime, then the p^r-fold branched covers of K and J, denoted by $\Sigma_{p^r}(K)$ and $\Sigma_{p^r}(J)$ respectively, are rationally homology cobordant. These two facts have been used quite successfully to obstruct knots from being slice and one could ask how good these obstructions are. One group tried to extend Piccirillo's construction. Given any finite set of integers $\{a_i\}$, she has a construction of a non-slice knot K where $S^3_{a_i}(K)$ is homology cobordant to $S^3_{a_i}(U)$ for all i, where U is the unknot. For the second problem, this group found an example of non-slice knot K where $\Sigma_{p^r}(K)$ bounds a rational homology ball for each prime p. The construction of this example is obtained by considering a 2-component link with a symmetry. The obstruction comes from the twisted Alexander polynomial.

Is every knot bounding a null-homologous disk in $\#_n\mathbb{C}$.
$\mathbb{P}^2$ *obtained from a slice knot by changing negative crossings to positive crossings?* A knot which is obtained from a slice knot by changing negative crossings to positive crossings bounds a null-homologous disk in a punctured $\#_n\mathbb{CP}^2$, for some n. Further, it was shown by Cochran and Tweedy [Cochran-Tweedy:2014-1] that a knot bounds a null-homologous disk in a punctured $\#_n\mathbb{CP}^2$ if and only if it is obtained from a slice knot by adding a generalized positive crossings (allowing more than two strands). Yet, it is not known if there is a difference between just allowing crossing changes versus allowing generalized crossing changes. A group tried to attack this question by using an obstruction due to Alishahi-Eftekhary in [Akram-Eftekhary:2018-1] and an obstruction due to Owens in [Owens:2010-1], but it turned out both obstructions do not distinguish a crossing change and a generalized crossing change.

Is $\mathcal{T}$.
(m) *abelian for $m \geq 2$?* Even though the set of links up to concordance does not have a natural group structure, the set of m-component string links up to concordance, denoted by $\mathcal{C}(m)$, has a nice group structure. Note that $\mathcal{C}(1)$ is naturally isomorphic to the knot concordance group $\mathcal{C}$, which is an abelian group. A natural question is whether $\mathcal{C}(m)$ is abelian for $m \geq 2$. This can be easily shown not to be true using classical invariants such as Milnor's invariants. On the other hand, if we consider the set of m-component string links which are topologically concordant to the trivial string link up to smooth concordance, denoted by $\mathcal{T}(m)$, we do not know if $\mathcal{T}(m)$ is abelian or not for $m \geq 2$. A group used a method called the covering link calculus introduced by Cochran and Orr [timCO1] and used

by Cha and Powell [ChaPow14] to show that the successive quotients of the bipolar filtration of $\mathcal{T}(m)$ for $m \geq 2$ are highly non-trivial.

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