

# CONFIGURATION SPACES OF GRAPHS

organized by

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## Workshop Summary

### Overview

The purpose of this workshop was threefold:

- (1) to provide a platform for mathematicians studying configuration spaces of graphs to share their tools, techniques, heuristics, computations, ideas, and conjectures;
- (2) to facilitate the establishment of a shared knowledge base and shared goals within the area; and
- (3) to encourage the development of a meaningful community based around this research interest.

We followed the ordinary AIM structure consisting of two talks each morning, a moderated problem session Monday afternoon, and small group activities the other afternoons.

### *Selected reports from group projects*

*Toward new and convenient presentations of fundamental groups.*

This group investigated presentations of the braid groups of a graph related to the Artin presentation of the classical braid group. Such presentations are desirable in pursuing a deeper investigation of group-theoretic properties of graph braid groups, they hold the possibility of recovering the embedding genus of a graph from its graph braid groups, and they are of interest from the point of view of mathematical physics. By the end of the week, the group had proven that the braid groups of a 3-connected planar graph admit presentations by “loop” and “star” classes, which, after quotienting by the normal closure of the loops, reduce to the classical presentations of the braid groups of the plane. Progress had also been made in recovering surface braid groups in the non-planar case.

*The mapping cone of a half-edge deletion.*

This group studied the mapping cone of the map induced on configuration spaces by an inclusion of graphs which is surjective away from a single half-edge. The results include

- a geometric description of this mapping cone as the suspension of a related configuration space,
- an explicit homotopy equivalence between the mapping cone and the suspension, and
- a clean and parsimonious long exact sequence in homology suitable for inductive arguments that can be built up on more complicated graphs from simpler graphs.

*Random processes on trees.*

This group investigated a random process involving a pair of particles moving between vertices of a graph. This exclusion process gives rise to random element in the homology of the configuration space of two points. In a suitable scaling limit, the distribution of a long

trajectory converges to a multidimensional normal distribution. Conjecturally, one recovers the graph up to homeomorphism from the covariance matrix of this distribution, at least in the case of a tree. Many small examples were studied via simulation, and an efficient method was developed for computing the covariance matrix. Special cases of the problem were understood completely, and certain weak versions of the conjecture were ruled out.

*Other groups.*

Other groups that met once or twice produced the following.

- (1) Multiple investigations related to non- $k$ -equals configurations on graphs. The unordered configurations turn out to be a restricted version of the configurations of graphs with sinks. The ordered version has many interesting combinatorial features; one cubical model for the ordered non-3-equals configurations in the interval appears to be in bijection with the cubical faces of the permutahedron.
- (2) The beginnings of a generalization to arbitrary degrees of an argument showing that the second homology of planar graph braid groups is generated by classes arising from disjoint unions of loops, stars, and theta graphs with four edges.
- (3) Investigations of the cup product resulting in the confirmation of a conjectural description in the Swiatkowski model of the configuration space. This description will undoubtedly prove valuable in future computations.
- (4) The beginnings of a concrete description of the Leray spectral sequence for the inclusion of the ordered configuration space into the Cartesian product. This description seems to parallel the classical case of the configuration spaces of a manifold, expressing a decomposition into terms coming solely from the background space and terms tied to the combinatorics of partitions.