Crouzeix's conjecture

organized by
Mark Embree, Anne Greenbaum, and Michael Overton

Workshop Summary

Let $A \in \mathbb{C}^{n\times n}$ be a square matrix, and let $W(A)$ denote the numerical range of A,

$$W(A) = \{(Au, u) : \|u\| = 1\}.$$

In landmark 2007 paper, Michel Crouzeix proved that there exists a constant C such that

$$\|f(A)\| \leq C \max_{z\in W(A)} |f(z)|$$

for all functions f analytic on $W(A)$. (Here $(\cdot,\cdot)$ denotes an inner product, with $\|\cdot\|$ denoting the associated vector norm and the matrix norm it induces.)

In this 2007 paper, Crouzeix proved that $2 \leq C \leq 11.08$; earlier he had conjectured that, in fact, $C = 2$. The past decade has seen the conjecture proved in a number of special cases, including normal matrices, matrices of dimension $n = 2$ (in fact all matrices with degree-2 minimum polynomial), nilpotent matrices of dimension $n = 3$, and matrices for which $W(A)$ is a disk. Extensive careful numerical calculations have not yielded a counterexample to Crouzeix's conjecture.

Early in 2017, Crouzeix and Cesar Palencia made a significant breakthrough, giving a straightforward proof that improves the upper bound $C \leq 11.08$ to $C \leq 1 + \sqrt{2}$. This result provided great motivation for this AIM Workshop on Crouzeix's Conjecture.

The workshop began with a careful explication of the Crouzeix–Palencia proof, with Michel Crouzeix and Cesar Palencia giving the opening talks on Monday. These presentations prepared the participants to examine the potential for extending the proof technique to yield $C = 2$. A key ingredient of the Crouzeix–Palencia proof is the matrix

$$g(A) = \frac{1}{2\pi \mathrm{i}} \int_{\partial W(A)} \overline{f(z)}(zI - A)^{-1}\,\mathrm{d}z,$$

for which one can show that

$$\|f(A) + g(A)^*\| \leq 2 \max_{z\in W(A)} |f(z)|.$$

On Tuesday Anne Greenbaum described recent work (with Trevor Caldwell and Kenan Li) that shows how the Crouzeix–Palencia bound can be adapted to other simply connected open sets Ω that contain the spectrum but need not contain all of $W(A)$. Michael Overton then spoke about the careful use of numerical optimization to investigate the conjecture, minimizing the Crouzeix ratio

$$\frac{\max_{z\in W(A)} |p(z)|}{\|p(A)\|}$$

over polynomials p of a fixed degree and matrices A of a fixed size. The function that actually minimizes the Crouzeix ratio for a given matrix A is known to have the form $B \circ \varphi$, where φ is a conformal mapping of $W(A)$ to the unit disk, and B is a Blaschke product of degree at most $n-1$. Some properties of the optimal B were also discussed.

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is known to have the form $B \circ \varphi$, where φ is a conformal mapping of $W(A)$ to the unit disk, and B is a Blaschke product of degree at most $n-1$. Overton illustrated the structure that the optimal B takes for several examples. These presentations were followed by a joint talk by Christer Glader and Mikael Kurula describing their work (with Mikael Lindstrom) showing that Crouzeix's conjecture holds for 3×3 matrices for which $W(A)$ is an ellipse centered at an eigenvalue of A.

Wednesday began with a talk by Pamela Gorkin about elliptical numerical ranges, unitary dilations, Blaschke products, and connections to classical geometry. Thomas Ransford then surveyed the history of mapping theorems for the numerical range, including early contributions by Berger, Pearcy, and Kato, through to contemporary results of Davidson, Paulsen, and Woerdeman. In particular, a result in a recent preprint of the last three authors establishes that Crouzeix's conjecture ($C = 2$) is equivalent to the bound

$$w(f(A)) \le C' \max_{z\in W(A)} |f(z)|$$

with $C' = 5/4$, where $w(\cdot)$ denotes the numerical radius (i.e., the largest magnitude of any point in the numerical range). (Corollary 3.2 of the Crouzeix–Palencia paper notes that $C = 1+\sqrt{2}$ implies $C' = \sqrt{2}$.)

The talks on Thursday morning were motivated by applications. Felix Schwenninger surveyed growth bound results for powers of matrices and operators, including the Kreiss matrix theorem and properties of Tadmor–Ritt operators. He also posed some related open questions. (For example, if $\|e^{tA}\|$ is bounded for all $t \ge 0$, must the same be true for $\|e^{tA^{-1}}\|$ when A is an operator on an infinite dimensional Hilbert space?) Jurjen Duintjer Tebbens discussed his surprising theorems (with Meurant) concerning the location of Arnoldi Ritz values. Such Ritz values are the eigenvalues of restrictions of A to lower-dimensional Krylov subspaces; in applications they are often used as estimates for the eigenvalues of A. Duintjer Tebbens showed how one can construct an A that has essentially any desired Ritz values and eigenvalues, implying that the Ritz eigenvalue estimates can be arbitrarily poor for some problems.

Elias Wegert opened Friday morning with a discussion of Nevanlinna–Pick interpolation and Riemann–Hilbert problems, along with some speculations about directions for future work on the numerical range. The workshop's final talk was delivered by Bernd Beckermann, who addressed the Zolotarev problem of optimal rational approximation, and its connection to Faber polynomials, optimal rational functions of matrices, and Sylvester matrix equations.

Throughout the week the afternoon problem sessions addressed a variety of topics, including an exploration of the Crouzeix–Palencia proof and its possible extensions, numerical computations of optimal Blaschke products, inverse numerical range problems, and barycentric interpolation approaches to Crouzeix's conjecture. One early outcome of the workshop is a collaborative preprint of Ransford and Schwenninger that provides a simplified proof of a key lemma in the Crouzeix–Palencia paper, and illustrates some limits to the proof technique used to establish the $1+\sqrt{2}$ constant; see https://arxiv.org/abs/1708.08633.

Schwenninger curated the list of problems that were proposed for study at the workshop, and are available from the American Institute of Mathematics website.