Problems for 2015 AIM workshop on

Dynamical Algebraic Combinatorics

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May 29, 2015

Contents

Note: For discussion of antichains, order ideals, and $J(P)$ (the set of order ideals of a poset P), see *Enumerative Combinatorics I* by Richard Stanley. For discussion of $O(P)$ (the order polytope of a poset P), see *Two Poset Polytopes* by Richard Stanley. For discussion of promotion and rowmotion, see *Promotion and Rowmotion* by Jessica Striker and Nathan Williams. For discussion of Panyushev complementation, see *On orbits of antichains of positive roots* by Dmitri Panyushev. For discussion of the concepts of homomesy and 0-mesy, see *Homomesy in products of two chains* by James Propp and Tom Roby. For discussion of piecewise linear and birational liftings of promotion and rowmotion, see *Piecewise-linear and birational toggling* by David Einstein and James Propp. For discussion of ASMs (alternating sign matrices) and Wieland's gyration action on ASMs, see *The many faces of alternating-sign matrices* by James Propp and *Promotion and Rowmotion* by Jessica Striker and Nathan Williams. For discussion of the CSP (cyclic sieving phenomenon) see *What is cyclic sieving?* by Victor Reiner, Dennis Stanton, and Dennis White.

1 Homomesies

1.1 Classifying homomesies

Let P be a finite poset and $T : J(P) \to J(P)$ some invertible action on the set of order ideals of P. Given an order ideal $I \in J(P)$ and an element $x \in P$, let $\mathbf{1}_x(I)$ be 1 or 0 according to whether or not $x \in I$, and let V be the real vector space of functions from $J(P)$ to $\mathbb{R}$ spanned by the function $\mathbf{1}_x$. Let V_0 be the subspace of V consisting of all 0-mesies, that is, functions $f : J(P) \to \mathbb{R}$ such that $\sum_{I\in O} f(I) = 0$ for every T-orbit O in $J(P)$.

Problem 1.1. *What is the dimension of V_0, and what is a natural basis for it?*

We know the answer when P is a product of two chains and T is rowmotion or promotion, but not for rowmotion and promotion of other posets.

One can ask the same question with antichains instead of order ideals. Again, we know the answer when P is a product of two chains and T is rowmotion or promotion, but not for the rowmotion and promotion actions on other posets.

1.2 Telescoping

Let T be an action on a set X all of whose orbits are finite, and let f be some real-valued 0-mesy of (X, T) (that is, a real-valued function on X whose average value on each T-orbit is 0, or equivalently, whose sum on each T-orbit is 0), The most straightforward way to prove that the sum of f over every T-orbit is zero is to find a way to exhibit a function g on X for which $f(x) = g(x) - g(T(x))$ for all $x \in X$; for in that case 0-mesy is just a matter of telescoping. And indeed, some of the known proofs of homomesy rely upon similar tricks. This prompts one to ask: To what extent can proofs of homomesy results (especially 0-mesy results) be simplified by finding constructions of such "integrals" g?

Note that for any 0-mesic function f there are typically many functions g satisfying the relation $f(x) = g(x) - g(T(x))$; the problem is the following.

Problem 1.2. *Find simple and uniform constructions for such functions g for a wide range of 0-mesies f.*

1.3 Dynamical closures

In most cases of combinatorial interest, the "feature space" V is not preserved by the map T; that is, for $f : X \to \mathbb{R}$ a function in V, the time-shifted function $f \circ T : X \to \mathbb{R}$ typically is not in V. It therefore seems natural, when T is of finite order (order n, say), to replace V by a larger but still finite-dimensional space V^T (the "dynamical closure" of V) defined as the smallest set of functions that contains V and contains $f \circ T$ whenever it contains f. Indeed, many important dynamical properties of (X, T, V) reduce to linear (or more generally affine) relations satisfied in V^T: *invariance* asserts that for some particular $f \in V$, f equals $f \circ T$; *homomesy* asserts that for some particular $f \in V$, $f + f \circ T + f \circ T^2 + \cdots + f \circ T^{n-1}$ equals a constant function; and *reciprocity* asserts that for some particular $f, g \in V$ and some particular k, $f + g \circ T^k$ is the zero function. Also (to give an example that seems to be very narrow but whose seeming narrowness may be merely a reflection of our ignorance of analogous behavior in other dynamical systems), we may note that the combinatorial fact that underlies the existence of the Armstrong-Stanley way of looking at Panyushev complementation of antichains in $[a] \times [b]$, namely, the fact that A contains an element in the ith fiber if and only if the Panyushev complement of A contains an element in the $(i+1)$st fiber (as long as $i < a$), can also be expressed as a linear relation in V^T.

Here is an outline for how to approach V^T systematically. Letting $f_1, \ldots, f_N$ denote a basis for V^T, define $V^T(X)$ to be the set of N-tuples $(f_1(x), \ldots, f_N(x))$ as x varies over X. Figuring out what linear relations are satisfied by the functions $f_1, \ldots, f_N$ when they are restricted to $V^T(X)$ is equivalent to:

Problem 1.3. Characterize the affine closure of $V^T(X)$ in V^T.

One could explore this computationally for specific dynamical systems (X, T), and infer patterns that would lead to conjectures. The cases to start with are rowmotion and promotion on $[a] \times [b]$, since a wealth of homomesy and reciprocity relations are known. Are there invariance relations as well, or other sorts of affine relations satisfied by $V^T(X)$ that don't follow from known homomesy and reciprocity relations?

2 Combinatorial Problems

2.1 The middle runner problem

Imagine n runners on a circular track, moving at various non-zero speeds (for simplicity, assume that the speeds are commensurable). The track has a starting line for counting laps (though runners are not required to start at the starting line), and the position of each runner at every instant is written as a number between 0 and 1 (representing how much of his/her current lap the runner has completed). At any instant t, let

$$p_1(t) \leq p_2(t) \leq \cdots \leq p_n(t)$$

be the sorted positions of the runners, and for $1 \leq i \leq n$ let $\overline{p_i}$ denote the average value of $p_i(t)$ over the course of one full period (the time it takes for all the runners to return to where they respectively started, which is just the lcm of the periods of all the runners).

Conjecture 2.1. *For $i + j = n + 1$, $\overline{p_i} + \overline{p_j} = 1$.*

The middle runner problem takes its name from the special case in which n is odd and $i = j = (n + 1)/2$: it says that the position of the middle runner is $1/2$ on average.

The case in which all runners have the same speed has been solved, but the general case remains open.

2.2 Cores

There are bijection between simultaneous (a, b)-cores, lattice paths in the triangular region with vertices $(0, 0), (a, 0), (a, b)$, and certain (a, b)-noncrossing partitions [AHJ14, ARW13]. There are $\frac{1}{a+b}\binom{a+b}{b}$ such objects.

Armstrong conjectured that the average size $|c|$ of an (a, b)-core c was

$$\frac{(a-1)(b-1)(a+b-1)}{24}.$$

This was proved in the Catalan case in [SZ13], the Fuss-Catalan case in [Agg14], and (very recently!) in full generality (using the polynomial method and Erhart theory—cores are naturally points in the root lattice in a dilation of the fundamental alcove in affine type A) in [Joh15]. Thus, the number and average size correspond to the 0th and 1st moments of the generating function $\sum_{c \text{ an } (a,b)-\text{core}} q^{|c|}$.

Problem 2.2. *Find and prove formulas for higher moments of cores:*

$$\sum_{c \text{ an } (a,b)-\text{core}} |c|^i.$$

One can rotate an (a, b)-noncrossing partition, which can be modeled using toggles on rational slope lattice paths.

Conjecture 2.3. *The set of (a,b)-noncrossing partitions under rotation exhibits the CSP using the polynomial $\frac{1}{[a+b]_q}\binom{a+b}{b}_q$.*

2.3 Perfect matchings

The Aztec diamond graph of order n has $2^{n(n+1)/2}$ perfect matchings (see e.g. [EKLP92a, EKLP92b]); fix n, and let X be the set of all such perfect matchings. for each edge e and each perfect matching M of the graph, let $1_e(M)$ be 1 if e belongs to M and 0 otherwise. Then the average value of $1_e(M)$ as M ranges over X can be interpreted as the probability that, if one chooses uniformly at random from the set X, the perfect matching one chooses will contain the edge e. These probabilities (as e ranges over the set of edges of the Aztec diamond graph of order n) are all rational numbers with denominators dividing $2^{n(n+1)/2}$, and there is a priori no reason to think that the edge-inclusion probabilities (that is, the probabilities that specific edges will appear in a uniformly random perfect matching) should be expressible as fractions with a much smaller denominator. However, it is known (though possibly not mentioned in published articles) that all such probabilities can be written as fractions with denominators dividing 2^n (which is on the order of the square root of $2^{n(n+1)/2}$).

Problem 2.4. *Is there a cyclic action of order 2^n on the set of perfect matchings of the Aztec diamond graph of order n, such that the edge-inclusion indicator functions associated with all the edges of the Aztec diamond graph are all homomesic?*

This line of thinking is inspired by Sam Hopkins' succinct formulation of the homomesy enterprise via the slogan "Small denominators are explained by group actions."

A possible avenue to pursue in solving Problem 2.4 may be an analysis of domino shuffling, since the set of Aztec diamonds is naturally divided into "equivalence classes" of size 2^n by the shuffling algorithm.

2.4 Resonance

Some researchers have recently been studying combinatorial actions that are not strictly speaking of finite order, or at least not uniformly finite as some size parameter n varies, but still exhibit some forms of periodicity. An important example is Wieland's gyration operation on Alternating Sign Matrices (ASMs). When n is small, the $2n$th power of the gyration operation on n-by-n ASMs is the identity map, so that all orbits have size dividing $2n$, but as n gets larger this ceases to be the case. Instead one finds orbits whose sizes are "mostly" multiple of $2n$, or submultiples $k(2n)/m$ where m is a small divisor of $2n$. This is a fairly squishy notion, for what do "mostly" and "small" mean? Without having answers to these questions, we have charged forward and dubbed this "resonance"; part of the challenge here is making a good definition of the phenomenon being studied.

Problem 2.5. *What is resonance? And why are there systems that exhibit it?*

Another example of resonance appears with regard to rowmotion on order ideals of posets of the form $[a] \times [b] \times [c]$ once a, b, and c are all large enough.

One attempt to understand resonance has looked at the *piecewise-linear (PL)* lifts of maps that at the combinatorial level exhibit resonance. In many cases, we find that the PL dynamical system exhibits a whole spectrum of periods, each associated with a positive-measure subset of the polytope on which the map is acting. (For instance, for ASMs of order 4, the PL lift of gyration has lots of orbits of size 8, and lots of orbits of size 24, and many of much larger size, though curiously there are essentially none of size 16.) It would be good to understand this phenomenon better, and to find ways to relate it to the original, vaguely defined notion of resonance which pertains to orbits involving vertices of these polytopes, rather than interior points. This leads to the following problem.

Problem 2.6. *Find/define analogues of resonance phenomena for the PL dynamical systems mentioned above and their birational lifts.*

For background on piecewise-linear and birational lifts of toggle-group actions, see [EPb] and the more detailed article-in-progress [EPa].

2.5 Undiscovered combinatorial models

2.5.1 The $3n - 2$ Problem

The fact that Wieland's gyration operation T on n-by-n ASMs "resonates with $2n$" (even though it is not periodic with period $2n$) can be readily understood in terms of the Fully Packed Loops model and the link pattern associated with an ASM (see [Pro01]); these link patterns admit a natural rotation action of order $2n$, and this action is compatible with gyration in the sense that turning an ASM into a link pattern and then applying rotation gives the same outcome as first applying gyration and then turning the resulting ASM into a link pattern.

Striker and Williams found another action on ASMs, called *superpromotion* in [SW12], that resonates with $3n - 2$. This suggests that there may be a map from n-by-n ASMs to some other class of combinatorial objects that admits a natural cyclic group action of order $3n - 2$.

Problem 2.7. *What action of order $3n - 2$, on combinatorial objects of some unspecified kind, plays the role that rotation of link-patterns does in the case of gyration?*

One possible approach would be to encode ASMs of order n as certain points in $k\mathcal{O}(\Phi^+(A_n))$. The vertices $k\mathcal{O}(\Phi^+(A_n))$ are naturally labeled by noncrossing partitions: on such an encoding, is the link pattern uncovered by projecting an ASM to a vertex of $k\mathcal{O}(\Phi^+(A_n))$? If so, the vertices are also labeled by triangulations.

Nathan Williams has pointed out that $n + 2$ (the order of rotation of a triangulation), $2n$ (the order of the Kreweras complement), and $3n - 2$ (the order of this conjectural mystery action) are in arithmetic progression.

2.5.2 Multi-noncrossing models

One can construct multi-noncrossing objects using a subword construction due to Ceballos, Labbé, Stump [CLS14].

These have a well-known triangulation model, which was studied, for example, by Pilaud in [PP12]. In types $A, B, H_3, I_2(m)$, these also have a nonnesting model as P-partitions in the root poset.

Problem 2.8 ([CLS14])**.** *Prove a cyclic sieving phenomenon for multitriangulations, using the analogue of Cambrian rotation and the polynomial*

$$\prod_{0\leq j<k}\prod_{1\leq i\leq n}\frac{[d_i+h+2j]_q}{[d_i+2j]_q}.$$

Problem 2.9. *Is there a corresponding multi-noncrossing partition model?*

One would expect that this has something to do with the root configuration, and the case $k=1$ is well-known in classical types. Using the correspondence between the type B_n root poset and $[n]\times[n]$, the case $k=2$ for the type B_n root poset corresponds to certain Narayana numbers [SW12]. Similarly, the case $k=3$ corresponds to Baxter numbers [Dil14]. Both of these have noncrossing models: Narayana numbers are well-known to be noncrossing partitions with a specified number of blocks; N. Reading has given a combinatorial description of Baxter permutations as noncrossing diagrams on a horizontal line of points using arcs that stay above or below the points [Rea14].

2.6 Products of chains

Bloom et al. [BPS13] proved a homomesy result for rectangular semistandard tableaux under promotion. But we also know that promotion of semistandard tableaux is a special case of PL promotion in the order polytope of a product of two chains (see http://jamespropp.org/gtt-promotion.txt), and we also know homomesy results for rowmotion and promotion in products of two chains (Propp and Roby).

Problem 2.10. *How do the results of Bloom, Pechenik, and Saracino relate to the results of Propp and Roby?*

Also, there is work to be done regarding rowmotion on products of three chains. The obvious cardinality statistic is not in general homomesic under rowmotion in general, but various other statistics are (or at least appear to be, experimentally).

Problem 2.11. *What are the homomesies of rowmotion acting order ideals in a product of three chains? Likewise, what are the homomesies of Panyushev complementation acting on antichains in a product of three chains?*

3 Coxeter-theoretic Problems

3.1 Bijactions in Cataland

The following problem is given in greater detail in [Wil14] and in much greater detail in [Wil13].

Let W be a finite Weyl group (or type H_3 or $I_2(m)$) and let c be a Coxeter element. Fix the word in simple reflections $\mathbf{Q} = (\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_{N+n}) := \mathbf{c}\mathbf{w_o}(\mathbf{c})$ (here $w_o(c)$ is the c-sorting word for w_o). The (W, c)-subwords are the elements of the set

$$\mathsf{Asoc}(W, c) := \{(i_1 \leq i_2 \leq \cdots \leq i_N) : \mathbf{Q}_{i_1}\mathbf{Q}_{i_2}\cdots\mathbf{Q}_{i_N} = w_o\}.$$

We find it convenient to complete the word $\mathbf{Q}$ to the word $\mathbf{Q}\overline{\mathbf{Q}} = \mathbf{c}^{h+2}$ (up to commutations), and to think of a subword in $\mathsf{Asoc}(W, c)$ as a doubled subword of this doubled word.

Define the *nonnesting c-Cambrian rotation* $\mathsf{Camb}_c : \mathsf{NN}(W) \to \mathsf{NN}(W)$ by

$$\mathsf{Camb}_c := \mathsf{Tog}_{\mathrm{inv}(\mathbf{w_o}(\mathbf{c}))}\mathsf{Tog}^+_{\mathrm{inv}(\mathbf{w_o}(\mathbf{c}))},$$

where $\mathsf{Tog}^+_\alpha(x) := \begin{cases} \mathsf{Tog}_\alpha(x) & \text{if } \alpha \notin \Delta(W); \\ x & \text{otherwise,} \end{cases}$ and $\mathsf{Tog}^+_{\alpha_1\alpha_2\cdots\alpha_i} := \mathsf{Tog}^+_{\alpha_i}\cdots\mathsf{Tog}^+_{\alpha_1}$.

Conjecture 3.1. *A bijaction*[1] *from $J(\Phi^+(W))$ under Camb_c to $\mathsf{Asoc}(W, c)$ (under noncrossing Cambrian rotation) is given as follows. Beginning with a nonnesting partition x, compute the orbit*

$$\left(x, \mathsf{Camb}_c(x), \mathsf{Camb}^2_c(x), \ldots, \mathsf{Camb}^{h+1}_c(x)\right).$$

The subword of $\mathbf{c}^{h+2}$ is given by replacing each nonnesting partition $\mathsf{Camb}^k_c(x)$ in this sequence by a copy of $\mathbf{c}$, adding to the subword those simple reflections whose corresponding roots are in $\mathsf{Camb}^k_c(x)$.

This has immediate homomesy implications—for example, in type A_n, this corresponds to homomesies of rotation of a triangulation.

There is so much more to say on this: there is a similar and intimately related map for the Kreweras complement to noncrossing partitions; one can walk on the Cambrian lattice in this order to realize the noncrossing versions.

3.2 Nonnesting Cataland Lifts

Birational toggles in $\mathbf{w_o}(\mathbf{c})$ root orders appear to continue to have order $2h$ in type A_n. This fails, for example, in type D_4 (as does birational rowmotion).

Conjecture 3.2. *Birational toggles in $\mathbf{w_o}(\mathbf{c})$ root orders have order $2h$ in the coincidental types $A, B, H_3, I_2(m)$.*

[1] Bijaction: a bijection induced by an action.

Can we give piecewise-linear and birational analogues of Armstrong, Stump, and Thomas' proof (see [AST13]) of Panyushev's Conjecture 2.1 (see [Pan09]) asserting homomesy of antichain cardinality under rowmotion? To clarify the meaning of this question we provide a bit of background.

Let P be the root poset of type A_n, with order polytope $\mathcal{O}(P)$ and chain polytope $\mathcal{C}(P)$. Let $\rho : \mathcal{O}(P) \to \mathcal{O}(P)$ be PL rowmotion and $\phi : \mathcal{O}(P) \to \mathcal{C}(P)$ be Stanley's transfer map [Sta86]. Then the map $\rho' := \phi \circ \rho \circ \phi^{-1} : \mathcal{C}(P) \to \mathcal{C}(P)$ may be viewed as a PL analogue of the Panyushev complement (since its restriction to the vertices of $\mathcal{C}(P)$, that is, to the antichains of P, is Panyushev complementation). Let $f : \mathcal{C}(P) \to \mathbb{R}$ be the function that adds all the coordinates of a point in $\mathcal{C}(P)$ (the PL analogue of the cardinality of an antichain).

Conjecture 3.3. *f is homomesic under the action of ρ', with average value $n/2$.*

It appears that a similar homomesy holds for the natural birational lift of ρ'. Experiments in Mathematica show (by brute force) that $\widehat{f}$ is 0-mesic under the action of $\widehat{\rho'}$ for the cases $n \leq 3$, where $\widehat{f}$ is the logarithm of the product of the entries of a triangular array of formal indeterminates, and $\widehat{\rho'}$ is the birational lift of ρ' defined in the most straightforward fashion.

A birational Armstrong-Stump-Thomas theorem would yield the "classical" Armstrong-Stump-Thomas result as a corollary in the usual way (first tropicalize to obtain the PL version, then specialize to the vertices of the order polytope).

It should also be noted that Panyushev's article contains other conjectures about homomesy for cardinality of antichains, which apparently have not been proved.

3.3 Coincidental Types

This problem is taken from [Wil13, Wil14].

Define the posets $\square_n := [n] \times [n]$, $\lrcorner_n = \mathcal{J}([2] \times [n])$, and $\searrow_n := \mathcal{J}^n([2] \times [2])$. These are the (Gaussian/minuscule) root posets for certain maximal parabolic quotients W^J [Ste96].

Theorem 3.4. *We have the following equalities:*

$$\begin{aligned}
|\mathcal{L}(\Phi^+(A_n))| &= 2^{n(n-1)/2}|\mathcal{L}(\lrcorner_n)|, & 2^n|\mathcal{J}(\Phi^+(A_n) \times [k])| &= |\mathcal{J}(\lrcorner_n \times [2k+1])|;\\
|\mathcal{L}(\Phi^+(B_n))| &= |\mathcal{L}(\square_n)|, & |\mathcal{J}(\Phi^+(B_n) \times [k])| &= |\mathcal{J}(\square_n \times [k])|;\\
|\mathcal{L}(\Phi^+(H_3))| &= |\mathcal{L}(\lrcorner_5)|, & |\mathcal{J}(\Phi^+(H_3) \times [k])| &= |\mathcal{J}(\lrcorner_5 \times [k])|; \text{ and}\\
|\mathcal{L}(\Phi^+(I_2(2m)))| &= |\mathcal{L}(\searrow_{m-2})|, & |\mathcal{J}(\Phi^+(I_2(2m)) \times [k])| &= |\mathcal{J}(\searrow_{m-2} \times [k])| \text{ for } m \geq 2.
\end{aligned}$$

Problem 3.5. *Give combinatorial proofs of the equalities above.*

We will refer to an equation in Theorem 3.4 by its row (A, B, H, or I) and its column ($\mathcal{L}$ or $\mathcal{J}$). Note that $A\mathcal{J}$ is already interesting for $k = 1$.

Problem 3.6. *Relate both sides of the $\mathcal{L}$ identities under promotion, and both sides of the $\mathcal{J}$ identities under (birational) promotion/rowmotion.*

The hook-length and shifted hook-length formulas prove $A\mathcal{L}$ and $B\mathcal{L}$. R. Proctor simultaneously established $B\mathcal{L}$ and $B\mathcal{J}$ with a representation-theoretic proof of $B\mathcal{J}$ [Pro83], while K. Purbhoo in unpublished work and M. Haiman in [Hai92] found beautiful *jeu-de-taquin* bijections for $A\mathcal{L}$ and $B\mathcal{L}$, respectively. I believe that the remaining equalities are new or trivial. [2]

3.4 Hurwitz Actions on Factorizations of c

This problem comes from [Wil13]; D. Bessis made a reference to the possible existence of such a problem in Bielefeld (`http://www.math.uni-bielefeld.de/birep/meetings/ncp2014/`).

For W a Coxeter group with degrees $d_1, d_2, \ldots, d_n$ and Coxeter number h, the factorizations of a Coxeter element c are counted by the uniform formula

$$|\mathrm{Red}_T(c)| = \frac{n!h^n}{|W|} = \prod_{i=1}^{n} \frac{ih}{d_i}.$$

In type A_n, these are equinumerous with parking functions, so it is possible to rephrase this problem in type A in terms of parking functions; there is a simple bijection for linear c due to Stanley (a similar idea also works in types B and D).

We act on T-words (words using reflections T) using the *dual braid* move $\mathfrak{T}_i : \mathrm{Red}_T(w) \to \mathrm{Red}_T(w)$ by

$$\mathfrak{T}_i\Big(t_1, \ldots, t_i, t_{i+1}, \ldots, t_\ell\Big) = \Big(t_1, \ldots, t_{i+1}, (t_{i+1}t_it_{i+1}), \ldots, t_\ell\Big).$$

Fix W and a reduced word $\mathbf{w} = s_{i_1} \cdots s_{i_k}$ for w in A_{n-1}. Define the action

$$\mathfrak{T}_w := \mathfrak{T}_{i_1} \cdots \mathfrak{T}_{i_k}.$$

It is easy to see $\mathfrak{T}_w$ does not depend on the choice of reduced word for w. One can compute that $\mathfrak{T}_{w_o}$ has order $2h$ on $\mathrm{Red}_T(c)$, and that $\mathfrak{T}_c$ has order nh on $\mathrm{Red}_T(c)$.

Conjecture 3.7. *For $w = w_o, c$,*

$$\left(\mathrm{Red}_T(c), \prod_{i=1}^{n} \frac{[ih]_q}{[d_i]_q}, \mathfrak{T}_w\right)$$

exhibits the CSP.

[2] There is a simple representation-theoretic proof of $A\mathcal{J}$, but I don't know of any such proof for $H\mathcal{J}$.

This can probably be proved pretty easily for $w = w_o$ using a combinatorial construction that associates pairs of factors in the orbit under $\mathfrak{T}_{w_o}$ to certain bicolored quadrangulations.

There is an action of order h on $\mathrm{Red}_T(c)$, which is quite simply conjugation by c. It is easy to see that this gives all orbits of size h.

Problem 3.8. *The polynomial $\prod_{i=1}^{n} \frac{[ih]_q}{[d_i]_q}$ appears to propose orbit sizes for other multiples of h between h and nh when corresponding roots of unity are plugged in—can the conjecture above be generalized by describing the corresponding actions $\mathfrak{T}_w$?*

These elements are presumably related to solving $w^p = c^n$ in the braid group of type A_{n-1}, where n is the rank of W. For example, $(w_o)^p = c^n$ for $p = 2$, so $\mathfrak{T}_{w_o}$ gives an order $ph = 2h$ action; similarly, $c^p = c^n$ for $p = n$, so $\mathfrak{T}_c$ gives an order $ph = nh$ action.

4 Piecewise-Linear and Birational Toggles

4.1 Order polytope promotion and rowmotion

Problem 4.1. *Explicitly describe the decomposition of $\mathcal{O}(P)$ under piecewise linear promotion/rowmotion.*

Of particular interest would be to do this for $[a] \times [b]$ and the type B_n root poset [Pro83, Ste86].

The case $P = [2] \times [n]$ has been settled (the polytope is divided into "Catalan-many" simplices such that every power of the map, restricted to any particular simplex, is a linear map), but the case $P = [3] \times [n]$ is more complicated and has not been resolved.

4.2 Birational rowmotion on G/P

Problem 4.2. *Generalize Grinberg and Roby's proof of periodicity of birational rowmotion on rectangles uniformly to G/P.*

Grinberg and Roby's proof corresponds to the case when $G = GL(n)$ and P is a maximal parabolic subgroup—their coordinates appear to be related to its Plücker embedding, which has generalizations for other quotients (see, for example, [Hil82, page 184] or [FZ00, Section 3.1]. When P is minuscule, this would be a birational generalization of [RS13]. See also [RSW04].

Problem 4.3. *Study birational rowmotion on Proctor's d-complete posets.*

4.3 When is birational rowmotion periodic?

Grinberg and Roby show that birational rowmotion has finite order (i.e., is periodic) for a variety of graded posets of interest, but in general this appears to be the exception rather than the rule. When it holds, periodicity also follows (by tropicalization) for piecewise-linear rowmotion on the corresponding order polytopes, and generally one finds that the order of combinatorial rowmotion on the poset itself has much smaller order than one might naively expect.

We currently know that birational rowmotion is periodic for the following posets:

- the poset $[p] \times [q]$ which is the product of two chains, with order $p+q$.
- triangular posets created by cutting $[p] \times [p]$ square in half either vertically or horizontally, generally getting order $2p$.
- the class of *skeletal posets*, which generalize the class of graded forests. These are built up inductively by successively "grafting" multiple antichains above or below an existing poset, or by taking disjoint unions of graded skeletal posets of the same rank. The order can be easily bounded and computed algorithmically.

On the other hand, birational rowmotion (over fields of characteristic zero) has infinite order for the following simple examples:

- If P is the poset $\{x_1, x_2, x_3, x_4, x_5\}$ with relations $x_1 < x_3$, $x_1 < x_4$, $x_1 < x_5$, $x_2 < x_4$ and $x_2 < x_5$ (this is a 5-element 2-graded poset), then $\operatorname{ord}(R_P) = \infty$.
- If P is the "chain-link fence"
- If P is the Boolean lattice $[2] \times [2] \times [2]$, then $\operatorname{ord}(R_P) = \infty$.

We conjecture that birational rowmotion has order p for triangular posets made by cutting a $[p] \times [p]$ square into quarters, the two distinct cases being the "northeast" and "southeast" corners. We can show it holds for p odd, but the case of even p remains open (though all the evidence suggests that it's true). A generalization of this for "trapezoids" due to N. Williams is as follows:

Conjecture 4.4. *Let p be an integer > 1, and $s \in \mathbb{N}$. Let $\mathrm{NEtri}'(p)$ be the poset $\{(i,k) \in [p] \times [p] \mid i \leq k;\ i+k > p+1;\ \text{ and } k \geq s\}$ Then, $\operatorname{ord}\left(R_{\mathrm{NEtri}'(p)}\right) \mid p$.*

In general it seems that birational rowmotion has finite order for posets related to root systems, so there are several general classes that could be studied separately, or perhaps treated in a uniform way. For pictures and further details about all of this, the most complete and up-to-date source to consult is § 18–21 of `http://web.mit.edu/~darij/www/algebra/skeletal.pdf`. A concise sketch of the ideas involved is available in the twelve-page extended abstract for FPSAC 2014 [GR14].

4.4 Order polytopes and P-partitions

If we dilate the order polytope $\mathcal{O}(P)$ of a poset P by a factor of k, then the integer points of $k\mathcal{O}(P)$ are in bijection with P-partitions of height k, or—equivalently—$J(P\times[k])$. The usual piecewise linear toggles on $\mathcal{O}(P)$ now induce a toggle operation on these P-partitions.

(For P of tableaux shape with boxes $p \in P$, we record the number of elements (p, j) in the box p, and we may then add i to the boxes in the ith row to get column-strict tableaux.)

For certain posets (minuscule, types $A, B, H_3, I_2(m)$), there are very nice formulas for the number of these plane partitions. (Since they have hook-length formulas, we expect that there must also be nice formulas for P-partitions of height k in d-complete posets). For example, minuscule posets $\mathcal{P}$ have P-partitions of height k counted by

$$J(\mathcal{P} \times [k]) = \prod_{x\in P} \frac{[\mathsf{ht}(x) + k]_q}{[\mathsf{ht}(x)]_q},$$

while types $W = A, B, H_3, I_2(m)$ have the "uniform" formula [CLS14]

$$J(\Phi^+(W) \times [k]) = \prod_{0\le j<k} \prod_{1\le i\le n} \frac{[d_i + h + 2j]_q}{[d_i + 2j]_q}.$$

Problem 4.5. *When P is minuscule or coincidental, is there a cyclic sieving phenomenon the integer points of $k\mathcal{O}(P)$ under the induced actions of promotion/rowmotion?*

In the case of the root poset of A_n, there is a statistic—generalizing the major index—such that $J(\Phi^+(A_n) \times [k])$ is the weight-generating function for this statistic. Specifically, given a P-partition (where P is the root poset for A_n) whose entries lie between 0 and k, create a larger triangular array by sticking a row of k's at the bottom, then apply Stanley's transfer map to turn this into a point x in the chain polytope, with coordinates x_1 through x_p; the weight of the original P-partition can then be defined as q to the power of $\lambda(x)$, where λ is the linear form that weights entries in the jth column of the triangular array by $j - n - 1$ (for $1 \le j \le 2n + 1$). This weight doesn't just give a nice formula for the sum of the weights of the P-partitions of ceiling k, for each individual k; it does so in a uniform way (as in Chapoton's q-Ehrhart theory). Perhaps we should not be separating into cases according to k, but should be treating all k's together, by letting the cyclic group act on the a cone containing infinitely many points?

4.5 Cluster algebras and birational toggling

Cluster algebras have flips that change variables by acting on the Dynkin diagram (of simple roots). Birational toggles change variables by acting on the root poset (of all positive roots). For finite Weyl groups, there are wonderful dualities between the set of simple roots S and the set of all roots T. For example, $2|T| = h|S|$—for more information, see [Bes03].

Problem 4.6. *Make the analogy between birational toggles and cluster flips explicit.*

Presumably, this should fit into the S vs. T duality mentioned above. A starting point is the question of whether there is an analogue of the bijection between cluster variables and almost positive roots on iterates of rowmotion.

Perhaps frieze patterns (of both the PL and birational sort) would be a fruitful place to start. In the simplest non-trivial case, the dynamics of shifting the frieze pattern is essentially the dynamics of Lyness 5-cycles, as is briefly described in section 2.6 of (the November 2014 version of) Propp and Roby's "Homomesy in products of two chains" (arXiv:1310.5201v5).

4.6 Gelfand-Tsetlin triangles

Kirillov and Berenstein (see `math.uoregon.edu/~arkadiy/bk1.pdf`) describe what in modern parlance would be called a toggle-action on Gelfand-Tsetlin triangles. Such triangles may be viewed as lattice points in the order polytope of a certain poset, and Kirillov and Berenstein's involutions were the prototypical examples of fiber-flipping. As those authors noted, one can define an operation equivalent to Schützenberger promotion by taking appropriate products of these involutions; this gives rise to a cyclic group action on Gelfand-Tsetlin triangles whose properties deserve study (and indeed Grinberg has already proved a homomesy property of this action). However, a bigger finite group is only slightly offstage: the full symmetric group. (Joel Kamnitzer summarizes the construction of this action by saying "you take the cactus group of the root system and then quotient by the braid relations".)

Problem 4.7. *What are the homomesies of the symmetric group action on Gelfand-Tsetlin triangles?*

One might expect (at least naively) that there are more homomesies for the symmetric group action than for its cyclic subactions: making the group bigger means merging orbits, and this merging permits more averaging to take place.

Note that there is significant overlap with the question raised in Section **??**.

4.7 The birational toggle group

Let P be a finite poset. All the birational toggle operations taken together generate a group.

Problem 4.8. *When is the birational toggle group finitely presented?*

We know that this is the case when P is just a chain, for then the PL toggles are actual linear maps, and their birational lifts are just monomial maps. But even for as simple a poset as $[2] \times [2]$, we don't know the answer to this question.

Perhaps it is better to approach the birational toggle group from above. Certain algebraic combinations of the variables are invariant under all the birational toggles.

Problem 4.9. *Can we say what those combinations are? Can we then characterize the birational toggle group as being precisely the group of birational transformations that preserve those algebraic combinations?*

Apropos of invariance under toggling, we mention the case of the full toggle group on the poset $[2] \times [2]$. Of the 24 orders in which one can compose all 4 toggles, 8 are not conjugate to rowmotion or promotion or their inverses; indeed, the PL and birational versions of these compositions were shown by Einstein to be of infinite order. Nevertheless, there are things to be proved about the map (called "locomotion" for present purposes). For instance, the pictures at http://jamespropp.org/locomotion.pdf, which shows some projections of an orbit of locomotion strongly suggests that the orbit lies in a 2-dimensional surface in $\mathbb{R}^4$; that is, there are two conserved quantities, of which we know only one. Likewise, experimental studies suggest that other quantities are homomesic under locomotion, in an appropriately asymptotic sense of the word "average". (Similar, polyhedral pictures appear when one replaces birational locomotion by PL locomotion.)

Problem 4.10. *Is the PL toggle group isomorphic to the birational toggle group? Or is it a proper quotient group?*

5 Generalized Toggling

The following problems are associated to [Str15]. The main observation is that the toggle group need not be restricted to order ideals of a poset P. The particular structure of order ideals in a poset is unnecessary in the definition of the toggle group; the essential structure is merely that an order ideal is a subset of poset elements. Thus, given a finite ground set E, we can define a toggle group $T(\mathcal{L})$ on *any set of subsets* $\mathcal{L} \subseteq 2^E$.

Definition 5.1. Let E be a finite set and $\mathcal{L} \subseteq 2^E$. For each element $e \in E$ define its *toggle* $t_e : \mathcal{L} \to \mathcal{L}$ as follows.

$$t_e(X) = \begin{cases} X \cup \{e\} & \text{if } e \notin X \text{ and } X \cup \{e\} \in \mathcal{L} \\ X \setminus \{e\} & \text{if } e \in X \text{ and } X \setminus \{e\} \in \mathcal{L} \\ X & \text{otherwise} \end{cases}$$

Note that $t_e^2 = 1$ for all $e \in E$. We define the generalized toggle group as the group generated by these toggles.

Definition 5.2. Let $T(\mathcal{L})$ be the subgroup of the symmetric group $\mathfrak{S}_{\mathcal{L}}$, generated by $\{t_e \mid e \in E\}$. Call $T(\mathcal{L})$ the *toggle group* on $\mathcal{L}$.

Therefore, if we isolate any set of subsets $\mathcal{L}$ that has combinatorial meaning, we can use the toggle group to gain insight on these objects in ways similar to order ideals. Several examples of potentially interesting toggle groups are:

- Poset structures: chains, antichains, or interval-closed sets;
- More than one partial order on the same ground set;
- Graph structures: independent sets, acyclic subgraphs, vertex covers, edge covers, connected subgraphs;
- Matroids;
- Antimatroids.

5.1 Generalized toggling from the bottom up

Problem 5.3. *Explore these (and other) generalized toggle groups, look for homomesy and CSP, look at piecewise-linear and birational liftings.*

For example, one could look at toggling chains, then the piecewise-linear extension should be to the order complex. I have Sage code for generalized toggles and to search for homomesy in generalized toggle groups. Perhaps we could work on adapting Darij's birational code to the generalized lifted toggles at Sage Days.

5.2 Generalized toggling from the top down

Problem 5.4. *Start with a known action in the birational (or piecewise-linear) realm and find the corresponding generalized toggle group action in the combinatorial realm.*

One such example of a birational map is the pentagram map.

5.3 Subset toggling

Generalized subset-toggle groups are defined below. We could ask the same questions from the previous problem in this further-generalized context.

Let E be a countable set and $\mathcal{L} \subseteq 2^E$.

Definition 5.5. For any subset $S \subseteq E$ define its *(subset-)toggle* $t_S : \mathcal{L} \to \mathcal{L}$ as

$$t_S(X) = \begin{cases} X \triangle S & \text{if } X \triangle S \in \mathcal{L} \\ X & \text{otherwise} \end{cases}$$

where $X \triangle S$ denotes the symmetric difference of the sets X and S, that is, $X \triangle S = (X \setminus S) \cup (S \setminus X)$. We call $\{t_S \mid S \subseteq E\}$ the set of *(subset-)toggles.*

We define the power set toggle group as the group generated by all the (subset-)toggles on $\mathcal{L}$.

Definition 5.6. Let $T_{2^E}(\mathcal{L})$ be the subgroup of the symmetric group $\mathfrak{S}_{\mathcal{L}}$, generated by $\{t_S \mid S \subseteq E\}$. Call $T_{2^E}(\mathcal{L})$ the *power set toggle group* on $\mathcal{L}$.

One could also construct a toggle group using only some of the subset-toggles.

Definition 5.7. Let $\mathcal{K} \subseteq 2^E$. Define $T_{\mathcal{K}}(\mathcal{L})$ to be the subgroup of $T_{2^E}(\mathcal{L})$ generated by $\{t_S \mid S \in \mathcal{K}\}$. Call $T_{\mathcal{K}}(\mathcal{L})$ the *$\mathcal{K}$-toggle subgroup* on $\mathcal{L}$ (or generically, we call any $T_{\mathcal{K}}(\mathcal{L})$ a *subset-toggle group*).

5.4 Toggling noncrossing partitions

Given a noncrossing partition π of $\{1, 2, \ldots, n\} \in \mathsf{NC}(n)$ and $1 \leq i < j \leq n$, define τ' as follows:

- if i and j are consecutive elements of the same block B, *split* the block into two blocks (one consisting of all the elements of B that are $\leq i$ and the other consisting of all the elements of B that are $\geq j$) and leave all the other blocks alone;
- if i is the largest element of one block B_1 and j is the smallest element of another block B_2, *merge* the two blocks into one block $B_1 \cup B_2$ (as long as this will not violate the noncrossing condition) and leave all the other blocks alone;
- otherwise, do nothing.

We write $\tau' = \tau_{i,j}(\pi)$, and call the involution $\tau_{i,j} : \mathsf{NC}(n) \to \mathsf{NC}(n)$ *toggling at* (i, j).

Define the composite operation σ obtained by successively toggling at $(1, 2)$, $(2, 3), \ldots, (n-1, n), (1, 3), \ldots, (n-2, n), (1, 4), \ldots, (1, n-1), (2, n), (1, n)$.

Conjecture 5.8. *The number of blocks (a statistic on* $\mathsf{NC}(n)$*) is homomesic under* σ *with average value* $(n + 1)/2$ *on each* σ*-orbit.*

This conjecture has been verified for $n \leq 8$, with help from Striker's generalized toggling code (written in Sage). It should be noted that this action is not conjugate to the Panyushev complement, since for instance $\mathsf{NC}(4)$ consists of a single orbit of size 14 (rather than orbits of size 8, 4, and 2). In more detail, the orbit-decompositions for $2 \leq n \leq 8$ are $2 = 2$, $5 = 3 + 2$, $14 = 14$, $42 = 15 + 13 + 5 + 3 + 3 + 3$, $132 = 112 + 20$, $429 = 133 + 109 + 39 + 39 + 31 + 15 + 13 + 11 + 9 + 9 + 6 + 3 + 3 + 3 + 3 + 3$, and $1430 = 1240 + 144 + 32 + 8 + 6$.

Note that when n is even, the conjecture implies that all orbits have even cardinality. Also, if the conjecture is true, it's a dramatic illustration of the claim that homomesy can occur even when the orbit structure of a combinatorial dynamical system is "horrible" (from the point of view of cyclic sieving, say).

Variants of σ that arise from composing the $\tau_{i,j}$'s in a different order appear to be related to already-studied maps on Catalan objects.

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