Problems related to "Definability and Decidability Problems in Number Theory"

moderated by T. Scanlon, notes by J. Demeyer

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Question 1 (Shlapentokh, Eisenträger)**.** Let K be a number field.

(a) Does there exist a $(\forall x)(\exists \vec{y})$-formula $\theta_K(x)$ such that

$$\theta_K(a) \iff a \in \mathcal{O}_K?$$

(b) Same question with $\mathbb{Z}$ in place of $\mathcal{O}_K$.

(c) Does there exist a $\forall$-formula $\varphi_K(x)$ such that

$$\theta_K(a) \iff a \in \mathbb{Z}?$$

Question 2 (Pheidas)**.** Let K be a number field. Does there exist an $\exists$-formula $\delta(x, y)$ such that

$$\delta(a, b) \iff \text{(for every } p \text{ above a prime 1 mod 4)}(v_p(a) < 0 \ \rightarrow \ v_p(b) < 0).$$

This is known (more or less) for $K = \mathbb{Q}$ for primes 3 mod 4. A positive answer would imply existential definability of $\mathbb{Z}$ in $\mathbb{Q}$, hence a negative answer to HTP over $\mathbb{Q}$.

Question 3 (Demeyer)**.** A uniform statement of the theorem that r.e. sets are diophantine for $\mathbb{F}_p[t]$. How should we even state this precisely?

Question 4 (Pheidas)**.** Is $\mathbb{Q}$ existentially definable inside $\mathbb{Q}(t)$ (using the language of rings with t)? It is known to be definable (even without t in the language).

Using elliptic curves, one can easily existentially define *dense* subsets of $\mathbb{Q}$.

Question 5 (Koenigsmann)**.** Given $\mathbb{Q}^* \subseteq \mathbb{Q}^{**}$, both elementary equivalent to $\mathbb{Q}$ (in the language of rings). Does it imply that $(\mathbb{Q}^*)^{\mathrm{alg}} \cap \mathbb{Q}^{**} = \mathbb{Q}^*$ (i.e. is $\mathbb{Q}^*$ relatively algebraically closed inside $\mathbb{Q}^{**}$)?

If the answer is "NO" in some case, then we know that $\mathbb{Z}$ is not $\exists$-definable in $\mathbb{Q}$.

We know that $\mathbb{Q}^*$ is quadratically closed in $\mathbb{Q}^{**}$.

Question 6 (Koenigsmann)**.** Consider the language $\mathcal{O}_2 = \{0, 1, +, P_2\}$ where P_2 is the set of squares. Is every $\exists$-$\mathcal{O}_2$-definable set in $\mathbb{Q}$ already $\exists^+$-$\mathcal{O}_2$-definable?

Equivalently, are the following sets $\exists^+$-$\mathcal{O}_2$-definable:

- $\{x \in \mathbb{Q} \mid x \neq 0\}$,
- $\{x \in \mathbb{Q} \mid (\forall y)(y^2 \neq x)\}$.

Question 7 (Miller)**.** Given a countable graph (V, E), find a field K (of characteristic 0), polynomials $P \in \mathbb{Q}[x, t]$, $Q, R \in \mathbb{Q}[x, y, z]$ and a bijection $\alpha : V \to \{x \in K \mid (\exists t \in K)(P(x, t) = 0)\}$ such that for all $v, w \in V$:

1. $(v, w) \in E \;\leftrightarrow\; (\exists z)(R(\alpha(v), \alpha(w), z) = 0)$.
2. $(v, w) \notin E \;\leftrightarrow\; (\exists z)(Q(\alpha(v), \alpha(w), z) = 0)$.

Poonen: let P define a curve without non-trivial automorphism.

For the empty graph:

Find a polynomial $P \in \mathbb{Q}[x, t]$ such that K is a field generated by $x_1, x_2, \ldots, t_1, t_2, \ldots$ such that all $\{x_1, x_2, \ldots\}$ are algebraically independent and such that $P(x_i, t_i) = 0$ for all i and such that

1. For every permutation π of $\mathbb{N}$ there is a unique automorphism σ_π of K such that $\sigma_\pi(x_i) = x_{\pi(i)}$.
2. $\{x_i \mid i \in \mathbb{N}\} = \{y \in K \mid (\exists t)(P(y, t) = 0)\}$.

Question 8 (Pasten)**.** Do we know an r.e. subset of $\mathbb{Q}$ which is not diophantine? There is some evidence that $\mathbb{Z}$ is not diophantine, but we have no proof yet.

What is the "thinnest" subset of $\mathbb{Q}$ known to be diophantine?

Does Bombieri–Lang imply that the diophantine subsets of $\mathbb{Q}$ are either finite or have "fast" growth rates? This would give a different proof, assuming Bombieri–Lang, that $\mathbb{Z}$ is not diophantine.

Question 9 (Flenner)**.** Is there a sentence φ in the language of rings such that $\mathbb{Q}(t_1, \ldots, t_n)$ satisfies φ if and only if n is even?

Using ultrapowers, one might show this is not possible.

Question 10 (Pop)**.** For K and L finitely generated fields, does $K \equiv L$ imply $K \cong L$?

Question 11 (Flenner)**.** Does there exist a formula $\varphi(x, y)$ such that for any finitely generated field K and every rank one divisorial valuation v on K, there exists a parameter a such that $\mathcal{O}_v = \{b \in K \mid \varphi(b, a)\}$?

Question 12 (Videla)**.** Is there an existential analogue of Robinson's Q?

Question 13 (Vidaux)**.** Applications of exceptional sets for Büchi's problem for higher powers in characteristic p?

Question 14 (Pasten)**.** Existential undecidability of $K(t)$ in the language $\{0, 1, +, \cdot, T\}$ where $T(f) \leftrightarrow f \notin K$? This question is open for every field K.

Question 15 (Demeyer)**.** For various fields K, consider the relation on $K[t]$ defined by

$$R(f, g) \iff \deg(f) = \deg(g).$$

(a) Is this relation (existentially) definable for $K = \mathbb{C}$?

(b) For $K = \mathbb{F}_p$, is there an existential definition, uniform in p?

Question 16 (Miller)**.** Situate with respect to $\leq_T$ the following sets:

$$HTP_\infty(\mathbb{Q}), HTP_{\infty,1}(\mathbb{Q}), HTP_\infty(\mathbb{Z}), HTP_{\infty,1}(\mathbb{Z}).$$

(known: $0' \leq_T HTP_\infty(\mathbb{Z}) \leq_T 0''$)

Question 17 (Koenigsmann)**.** Decidability of $\mathbb{Q}^{\mathrm{solv}}$, $\mathbb{Q}^{\mathrm{ab}}$, $\mathbb{Z}^{\mathrm{solv}}$ and $\mathbb{Z}^{\mathrm{ab}}$.

Question 18 (Jarden)**.** Various questions from Jarden's talk.