Degree d points on algebraic surfaces

organized by
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Workshop Summary

Goals of the workshop

The workshop focused on degree d points on algebraic surfaces, with a view to developing foundations akin to what is well-established for the study of degree d points on curves and the study of measures of irrationality for higher dimensional varieties. The workshop participants consisted of researchers from a broad array of backgrounds (e.g., arithmetic of surfaces, geometric measures of irrationality, degree d points on curves, algebraic cycles, etc.), which gave a broad base of collective expertise for the working sessions.

The workshop began with two introductory lectures: one on degree d points on curves (by Isabel Vogt) and the second on measures of irrationality and their connection to low degree points on curves (by Nathan Chen). These talks established a common terminology and framework for many of the problems discussed over the week. Broadly, the questions and problems studied during the workshop fell into one of the following three themes.

(1) Let X be a smooth projective geometrically integral algebraic surface over a number field k. The density degree set $\delta(X/k)$ is the set of positive integers d for which the degree d points on X are Zariski dense in X. What structural properties does $\delta(X/k)$ have? For example, does it contain all sufficiently large integers if $X(k) \neq \emptyset$? Is it closed under multiplication by positive integers? Can it be stratified in terms of geometric constructions that can give rise to a Zariski dense set of degree d points? (Sections and)

(2) For a smooth projective geometrically integral curve C over a number field k, the Abel-Jacobi map $\mathrm{Sym}^d_C \to \mathrm{Pic}^d_C$ is a fundamental tool in understanding the degree d points. For a smooth projective geometrically integeral surface X, the analog should be the map $\mathrm{Hilb}^d_X \to \mathrm{Alb}^1(\mathrm{Hilb}^d_X)$. Are there classes of surfaces for which this map is well-behaved? For Fano surfaces, is it possible to explicitly describe or understand Hilb^d_X? (Sections , and)

(3) For a curve C/k, every integer in the Lüroth semigroup (i.e., the degrees of nonconstant map to $\mathbb{P}^1$) is in the density degree set. For a surface X, the degrees of nonconstant maps to $\mathbb{P}^2$ no longer form a semigroup, but the degrees that arise are still contained in the density degree set. For surfaces with negative Kodaira dimension or Kodaira dimension 0 over *nonclosed fields*, can we determine bounds on the minimum degrees of nonconstant maps to $\mathbb{P}^2$? (Over algebraically closed fields, optimal bounds are known for surfaces of negative Kodaira dimension and most abelian surfaces.) (Sections and)

Later in the week, there were a number of talks that discussed ideas and existing results that were relevant for the questions and problems discussed over the course of the workshop. Specifically, the talks covered

- cycle theoretic methods in measures of irrationality (Olivier Martin),
- Albanese varieties and torsors of symmetric products (Andres Fernandez Herrero and Jeff Achter),
- the structure of Chow groups of zero cycles over nonclosed fields (Valia Gazaki),
- Manin's conjecture, with a view towards Hilbert schemes of points on surfaces (Ulrich Derenthal),
- properties of Hilbert schemes of points on surfaces (Dori Bejleri),
- low degree points on modular curves (David Zureick-Brown), and
- stable birationality of Hilbert schemes of points (Morena Porzio).

Summaries of working groups

Here, we summarize several problems that were explored in the working groups.

Asymptotics and multiplicativity of the density degree set.

Let X be a smooth projective geometrically integral surface over a number field k. By covering X with curves, the working group showed that the density degree set $\delta(X/k)$ contains all sufficiently large multiples of the index of X. This property of the density degree set is philosophically related to whether the set of degrees of maps

$$\mathcal{S}(X) = \left\{ \delta > 0 \;\middle|\; \begin{array}{c} \exists \text{ dominant rational map} \\ X \dashrightarrow \mathbb{P}_k^{\dim X} \text{ of degree } \delta \end{array} \right\} \subseteq \mathbb{Z}_{>0}$$

contains all sufficiently large multiples of the index. The participants gave a positive answer to this question in the case that X contains at least three k-points.

The working group also considered the problem of whether $\delta(X/k)\mathbb{N} \subset \delta(X/k)$. The group showed that if there is a degree d rational map $X \dashrightarrow Y$ where Y is a surface that contains a Zariski dense set of rational curves, then $d\mathbb{N} \subset \delta(X/k)$. However, the question of whether $\delta(X/k)\mathbb{N} \subset \delta(X/k)$ remains open in general.

Density degree sets for products of curves.

Another working group studied the problem of describing the density degree sets $\delta(C \times D/k)$ of products of curves C/k and D/k. The group proved that there are always containments

$$\delta(C/k) \cdot \delta(D/k) \supset \delta(C \times D/k) \supset \delta(C/k) \cap \delta(D/k),$$

and produced many examples where the containments on either side are strict. For example, they constructed some examples of (pointless) degree 2 covers of $\mathbb{P}^1$ (with genus at least two), say X and Y, where the product $X \times Y$ does *not* have quadratic points. One can reduce to showing that the quotient $(C \times D)/\iota$ (by the diagonal involution) does not contain any rational points. For this, they wrote out explicit examples of equations for such a quotient surface which did not contain any rational points. As another example, the group showed that if C has genus 2 then the self-product can have a dense set of quadratic points coming from rational points on the symmetric square of C.

Points on the universal degree d.
hypersurface

One working group considered the problem of describing degree d points on the universal complete intersection variety of some multi-degree (over the function field of the coefficients). They observed that the problem is relatively easy for curves, but becomes interesting already for surfaces in $\mathbb{P}^3$. For instance, one might expect that cubic points on the universal cubic surface are all collinear. However, this cannot be farther from the truth: the group proved that cubic points on a cubic are always dense in Hilb^3. Along the way the group proved a useful lemma on the density of points on potentially unirational varieties, which might be of independent interest. For surfaces of higher degree d, some participants developed an approach to studying degree d points by combining a theorem of Mumford with some Cayley-Bacharach type theorems. In particular, they proved that for $d > 8$, noncollinear degree d points on the universal surface are not dense on the surface. The group conjectured that in fact there shouldn't be any noncollinear degree d points on this surface, and are continuing work in this direction.

Manin's conjecture for $Hilb^{[d]}$.
(X) and Cox rings The working group began by looking at the thesis of Le Rudulier, who looked at the asymptotic cardinality of the set of algebraic points of fixed degree and bounded height of a surface defined over a number field. The group then worked out the constants for the Batyrev-Manin-Peyre conjecture for Hilbert schemes of d points on del Pezzo surfaces. Later in the week, the group explored computing the Cox ring of the Hilbert schemes of d points on del Pezzo surfaces for some particular values of d and certain high degree del Pezzo surfaces.

Fibers of the Albanese map.

This group investigated some of the properties of the fibers of the Albanese map. The Albanese map from the Hilbert scheme of a surface factors through the map $\mathrm{Hilb}^d_X \to \mathrm{Sym}^d_X$ and it turns out that the Albanese of the symmetric product of a surface is equal to the Albanese of the original surface (this was the subject of Fernandez Herrero's and Achter's talks). For symmetric products of curves, the fibers of the Albanese morphism are Severi-Brauer varieties, but much less is known about the fibers of the Albanese map for symmetric products of surfaces. If X is a surface with Kodaira dimension at least 0, then it turns out that the general fiber F of the Albanese morphism will necessarily have Kodaira dimension at least 0. In studying this question, the group identified several papers of interest (papers of Mattuck, Huibregtse, Glazer-Hendel, etc.), and many participants are continuing to review the literature with arithmetic applications in mind.

Part of this working group focused on the specific example of the Fano surface of lines on a cubic threefold. By the work of many people (beginning with Clemens and Griffiths), much is known about the geometry of this surface. For instance, it is known that the Albanese of the surface is isomorphic to the intermediate Jacobian of the cubic threefold. Additionally, it is known that the Fano surface is of general type, and many irrationality results (over the complex numbers) have been computed by Gounelas and Kouvidakis. The group produced several interesting examples of low degree maps and points on this surface.

Abelian surfaces and del Pezzo surfaces: maps to $\mathbb{P}^2$*. and density degree sets*

This group explored the question of whether the existing geometric constructions for low degree maps from an abelian surface A to $\mathbb{P}^2$ (over the complex numbers) descend to a number field k. They determined that to descend the constructions, one needed to assume the abelian surface contained some 2-torsion points defined over k.

The group also explored the question of whether $2 \in \delta(A/k)$. They found that this is always true for products of elliptic curves - there is a paper showing that the associated Kummer surface for a product of elliptic curves has a Zariski dense set of quadratic points. Quadratic points are also dense on certain abelian surfaces since one can use multiplication by n to produce lots of quadratic points (starting from a non-torsion one, if it exists). The group spent some time thinking about abelian surfaces over k which are not isogeneous to the Jacobian of a genus 2 curve.

Quadratic points on degree 2 covers of $\mathbb{P}^2$*.*

This group investigated possible generalizations of the following result of Pasten–Vogt. *Assume Vojta's conjecture and let* $\pi\colon X \to \mathbb{P}^2$ *be a finite cover of degree* 2 *branched over a smooth curve of degree* $2m$ *with* $m \geq 6$*. Then there is a proper Zariski closed* $Z \subset X$ *such that all quadratic points in* X *outside of* Z *come from rational points on* $\mathbb{P}^2$*.*

Specifically, the group considered the problem of whether the hypothesis can be weakened to cover the case when π is just a *rational* map of degree 2, as opposed to a morphism. They spent some time thinking about how the ramification divisor would contribute to the canonical bundle and explored various ways to resolve or extend the rational map. The main challenge seemed to be quantifying the important condition that the ramification divisor has "sufficiently positive degree".

The group also explored constructions of surfaces that would show that the above theorem is sharp. A related question that came up is whether there exist integral surfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ whose class is $2(pt \times \mathbb{P}^2) + 2(\mathbb{P}^2 \times pt) + n(\text{line} \times \text{line})$ for large n? There is a paper of J. Huh that proves some integer multiple of this class is always represented by an integral surface. If taking a multiple is unnecessary, then such surfaces would suggest that the theorem above *cannot* be strengthened to apply to rational maps $\pi\colon X \dashrightarrow \mathbb{P}^2$.