Dynamical algebraic combinatorics

organized by
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Workshop Summary

Topics of the Workshop

This workshop centered around dynamical systems arising from algebraic combinatorics. Some well-known examples of actions on combinatorial objects are:

- promotion and evacuation for Young tableaux;
- the action of a Coxeter element on a parabolic quotient of a Coxeter group; and
- crystal operators on highest-weight representations.

Of particular relevance to this workshop were the actions and dynamical systems arising from:

- promotion and rowmotion for order ideals and antichains in posets; and
- their piecewise-linear and birational liftings.

One of the overall themes of this workshop was that well-behaved global actions (often cyclic of "small" order) arise from the composition of local actions (often involutions). A toy example is that the long cycle $c = (1, 2, \ldots, n)$ in the symmetric group $\mathfrak{S}_n$ may be factored as the product of transpositions

$$c = (1, 2)(2, 3) \cdots (n-1, n).$$

Similarly, the composition of Bender-Knuth involutions gives promotion of Young tableaux,, and the composition of poset toggles in linear extension order gives rowmotion of order ideals.

There are several frameworks now with which to study such actions. We will briefly review several of these frameworks here.

Cyclic Sieving.
One of the most basic problems in enumerative combinatorics is to count a given set of combinatorial objects. Having done this, we might wish to refine the enumeration and count our objects according to some interesting statistic. When we happen to have a cyclic action defined on our objects, we can also ask about the orbit structure. Remarkably, the enumeration according to a statistic and the orbit structure often turn out to be related.

The cyclic sieving phenomenon was introduced by V. Reiner, D. Stanton, and D. White as a generalization of J. Stembridge's $q = -1$ phenomenon.

Definition 0.1 (V. Reiner, D. Stanton, D. White)**.** Let X be a finite set, $X(q)$ a generating function for X, and let C_n the cyclic group of order n act on X. Then the triple $(X, X(q), C_n)$ exhibits the *cyclic sieving phenomenon* (CSP) if for $c \in C_n$,

$$X(\omega(c)) = \left|\{x \in X : c(x) = x\}\right|,$$

where $\omega : C_n \to \mathbb{C}$ is an isomorphism of C_n with the nth roots of unity.

Even though we can always artificially construct a polynomial to exhibit the CSP, much of the time a natural q-ification of an existing formula does the trick. The question then becomes to determine the statistic.

Homomesy.
It is often the case that statistics on a set of combinatorial objects with an invertible action have the property that the average value of the function on each orbit is the same for all orbits. This is formalized in J. Propp and T. Roby's homomesy phenomenon.

Definition 0.2 (J. Propp, T. Roby)**.** Given a set X, an invertible map $\tau : X \to X$, and a function $f : X \to \mathbb{R}$, the triple (X, τ, f) exhibits *homomesy* if there exists $a \in \mathbb{R}$ such that for all τ-orbits $O \subseteq X$,

$$\frac{1}{|O|} \sum_{x \in O} f(x) = a.$$

Bijactions and Resonance.

Definition 0.3. Let X be a finite set and let the cyclic group $C_n = \langle c \rangle$ act on X. Let $s : X \to S$ for some set S (we think of s as a statistic). Map $x \in X$ to the word $w(x)$ by

$$w(x) := s(x)s(c(x))s(c^2(x)) \cdots s(c^{n-1}(x)),$$

and take $W := \{w(x)x \in X\}$ to be the set of all such words. When w is injective, we call the equivariant bijection $w : X \to W$ a *bijaction*.

By construction, w takes X under c to the set W under left rotation. Such constructions have been used in several contexts to bijectively explain the existence of complicated cyclic actions. See also Section .

In general, the action of an invertible map τ on X results in unpredictable orbit structure. It can happen that this action nevertheless *resonates* with a small integer p as a pseudo-period, in the sense that most orbit-sizes are multiples of p (for example, alternating sign matrices under gyration). Such resonance phenomena can be explained in the language of Definition 0.3 using a *non-injective* map $w : X \to S$ that still takes X under τ to a set W under left rotation.

Goals of the Workshop

The main goals of the workshop were:

- to produce new combinatorial models that explain the existence of known cyclic actions and homomesies;
- to use data provided by cyclic actions, invariants, and homomesies to produce new bijections between combinatorial objects;
- to coordinate work on homomesy and generalized toggle group actions; and
- to suggest directions for future research.

Much of this area of combinatorics is still in early stages of exploration, with a number of tantalizing open problems whose statements require little background. Some examples of problems we are interested in are (for a full list, we refer the reader to https://cloud.sagemath.com/projects/
1d78-4e35-abe1-ae9fe9c403f2/files/Problem List.pdfthe workshop problem list):

- develop a combinatorial model of alternating sign matrices of size n that explains the existence of a cyclic action that appears to resonate with pseudo-period $3n-2$;
- uniformly prove that birational promotion and rowmotion have finite order on all minuscule posets;
- express known combinatorial actions as compositions of piecewise-linear involutions and investigate their birational analogues; and
- uniformly prove a bijection between nonnesting partitions and clusters related to Panyushev's homomesy conjectures.

Summary of Talks

There were two talks in the morning of each day of the workshop, except Thursday. The first four talks of the workshop served as introductory material to the various problems to be studied by the working groups. For a detailed account, we refer the reader to http://web.mit.edu/~shopkins/docs/aim_dyn_alg_comb_notes.pdfS. Hopkins' notes.

	Monday	Tuesday	Wednesday	Thursday	Friday
First Slot	J. Striker	J. Propp	D. Rush	G. Musiker	M. Glick
Second Slot	N. Williams	T. Roby	S. Okada	Problem Session	O. Pechenik

J. Striker - Toggle Group Actions, Applications, and Abstractions.

The first talk, given by J. Striker, gave an overview of the *toggle group* and its applications to various areas of interest, such as *cyclic sieving* and *homomesy*, as well as the *resonance* phenomenon (which includes the "$3n-2$" problem that was worked on later in the week). This talk also outlined how to abstract the toggle group from the world of posets to many other contexts, which is a perspective that J. Propp took in proposing the "noncrossing toggles" problem, which his working group tackled during the week.

N. Williams - Cataland.
The second talk, given by N. Williams, focused on problems arising from reflection groups. It is natural to interpret involutions (for example, the transpositions $(i, i+1) \in \mathfrak{S}_n$) as reflections—hence the appearance of reflection groups. Nathan emphasized two problems in his talk:

(1) The first problem was suggested by the following result of V. Reiner: the expected number of braid moves across all reduced words for the longest element w_o in $\mathcal{S}_n$ is exactly one. Based on computational evidence, Nathan presented the conjecture that, when restricting to the commutation class containing the reduced word $(s_1 \cdots s_n)(s_1 \cdots s_{n-1}) \cdots (s_1 s_2)(s_1)$, the expected number of braid moves is again one (note that this property is not true for arbitrary commutation classes). V. Reiner's original result may be phrased in terms of standard Young tableaux (SYT) of staircase shape; Nathan's problem may analogously be framed using shifted SYT of staircase shape.
(2) The second problem comes from W-Catalan combinatorics, and was necessarily of a more technical nature. The most frustrating gap in W-Catalan combinatorics is that there is currently no uniform proof that noncrossing W-Catalan objects are enumerated by the formula $\prod_{i=1}^{n} \frac{h+d_i}{d_i}$, where h is the Coxeter number of W, and

$d_1, d_2, \ldots, d_n$ are its degrees. [1] Briefly, the problem suggested an explicit *bijaction* between nonnesting partitions and facets of the subword complex, by proposing an analogue of Cambrian rotation on nonnesting partitions.

J. Propp - Dynamical Algebraic Combinatorics in the Combinatorial and Piecewise-Linear Realms.
The third talk, given by J. Propp, showed how the sorts of combinatorial maps discussed in the first two talks – bijections between finite sets – can often be generalized to piecewise-linear maps between polytopes. For instance, Schützenberger promotion on semistandard Young tableaux can be seen as the action of a piecewise-linear generalization of Striker-Williams promotion on the set of lattice points in order polytopes. Toggling in this context gets replaced by "fiber-flipping"; toggle groups (which in the combinatorial realm are necessarily finite) can be (and usually are) infinite; and the precise way in which finite orbits coexist with infinite orbits is quite different from what is normally seen in dynamical systems theory. The talk also stressed the notion that homomesy is not really about individual functions (often called "statistics") but rather about vector spaces of functions.

T. Roby - Birational Rowmotion.
The fourth talk, given by T. Roby, discussed the *birational* rowmotion map, ρ_B, focusing particularly on the periodicity question. Issues of homomesy are fairly easy to translate from the combinatorial and PL setting to the birational level: arithmetic means get replaced with geometric means, but questions of periodicity are harder. For certain kinds of generalized graded forests, called "skeletal posets", one can build up an inductive bound on the period of ρ_B. To prove that the order of ρ_B on a product of two chains $[p] \times [q]$ is $p + q$, more machinery was necessary, involving parameterizing the space of vertex-labelings of a poset by certain ratios of determinants. This is similar to Volkov's proof of the Zamolodchikov periodicity conjecture for Y-systems of type AA, and a reasonable problem (which was since been solved!) was to see whether one could find a more direct translation between birational rowmotion and Y-systems. More generally, this raises the question of a closer connection between this area of research and cluster algebras, birational RSK correspondence, etc.

The talks on the third day were given by D. Rush and S. Okada. D. Rush explained some results on cyclic sieving and homomesy in rowmotion on minuscule posets. He generalized the refined "file homomesies" proven by Propp and Roby for a product of two chains to an arbitrary minuscule poset in a natural way, and outlined a uniform proof. S. Okada talked about generalized parking functions, which relate to rational Catalan combinatorics. He gave an interesting open problem asking for a combinatorial interpretation of a q-analogue of the rational Catalan numbers in the non-coprime case.

The talks on the last two days, given by G. Musiker, M. Glick, and O. Pechenik, were related to results discovered during the week. They are discussed in Sections and .

Working Groups, Results, and Collaborations

In the afternoons of this workshop, there were groups working on the following topics (for a full list, we refer the reader to http://web.mit.edu/~shopkins/docs/aim_dyn_alg_comb_notes.pdfS. Hopkins' notes):

[1] This issue is to some extent resolved by the work by D. Armstrong, C. Stump, and H. Thomas, which answers D. Bessis and V. Reiner's refinement of a conjecture of D. Panyushev.

(1) the expected number of reduced words in a specific commutation class for $w_o \in \mathfrak{S}_n$;
(2) Cambrian rotation on nonnesting partitions;
(3) generalized toggling on noncrossing partitions;
(4) the relation between cluster algebras and birational toggling;
(5) the relation between promotion on increasing tableaux and plane partitions;
(6) the "$3n-2$" problem on alternating sign matrices and resonance for certain plane partitions;
(7) a probability measure on Young tableaux related to toggling; and
(8) a few other topics proposed by participants.

We outline below some progress and results obtained during the course of the workshop and shortly thereafter.

Problem (1).
Participants: Z. Hamaker, V. Reiner, A. Schilling, N. Thiery, G. White, and N. Williams.

This problem has been completely resolved in an unexpected way, using a bijection that takes as input a reduced word $\mathbf{w_o}$ in the fixed commutation class and a braid move in $\mathbf{w_o}$, and produces a new reduced word in the commutation class. The image of this map turns out to be all reduced words in the commutation class, and the expected number of braid moves is therefore 1. Since the end of the workshop, the group has determined that this bijection (when interpreted on tableaux) generalizes to a much larger class of objects.

Problem (2).
Participants: Z. Hamaker, A. Hicks, M. Thiel, N. Thiery, and N. Williams.

This group quickly determined that it would be desirable to use the conjectural Cambrian rotation (which we will now write Camb_c) on nonnesting partitions to produce a Cambrian recurrence on nonnesting partitions. In this way, nonnesting partitions would be treated no differently that W-noncrossing objects. Although the definition of Camb_c is given as a factorization of toggles in the root poset, the group made the bold assumption (as in the noncrossing case) that it would be possible to factor Camb_c as a composition $\mathrm{Camb}_{s_1} \circ \cdots \circ \mathrm{Camb}_{s_n}$ if $c = s_1 \cdots s_n$, where Camb_{s_i} was *independent* of c. The goal then became to write down each Camb_{s_i} as a product of toggles. Assuming that every reflection appeared at most once in Camb_{s_i}, the group wrote an algorithm so that the computer could determine the exact toggle sequences. The assumption that each toggle ought to appear once is false in general, but appears to be true in type A_n. In type A_n, they determined that Camb_{s_i} in fact is *dependent* on c. The group concluded the week by convincing themselves that they could write down explicit formulas for $\mathrm{Camb}^c_{s_i}$ as a product of toggles in type A_n.

Problem (3).
Participants: D. Einstein, M. Farber, E. Gunawan, M. Joseph, M. Macauley, J. Propp, and S. Rubinstein-Salzedo.

This group made significant progress at the workshop, showing experimentally that J. Propp's conjecture holds for a much broader class of actions, and that a larger vector space of functions is homomesic under all those actions. In the month following the workshop, the group continued to correspond by email, and was able to prove a much more general version of the original conjecture, situating it within the context of generalized toggling of independent sets in graphs.

Problem (4).
Participants: A. Berenstein, M. Glick, D. Grinberg, G. Musiker, and H. Thomas.

An important connection between birational rowmotion and cluster algebras was uncovered by this group. They showed that D. Grinberg and T. Roby's result that birational rowmotion on $[n] \times [k]$ has order $n+k$ is intimately related via an explicit construction to A. Zamolodchikov's periodicity conjecture that Y-systems of type $A_{n-1} \times A_{k-1}$ have order $h + h' = n + k$ (as proven by A. Volkov in type $A \times A$, and in full generality by B. Keller). Note that D. Grinberg and T. Roby's proof had been inspired by A. Volkov's proof, but the equivalence had not before been noted. G. Musiker and M. Glick spoke about this result on the last two days of the workshop.

This result raises the following dichotomy, which will need to be addressed in future work. The product of two chains $[n]\times[k]$ may be viewed as a product of the Dynkin diagrams of type A_n and A_k, but it also may be seen as the minusucule poset associated to the fundamental weight λ_k in type A_{n+k-1}. The result of the previous paragraph suggests that we should place periodicity of rowmotion results in the context of this first interpretation—that is, that we should view the product of two chains as a product of two Dynkin diagrams. But birational rowmotion has elegant periods for *all* minusucule posets, and does *not* appear to even have finite period for arbitrary products of Dynkin diagrams. This suggests that periodicity of rowmotion ought then to be seen in the second interpretation.

Problem (5).
Participants: K. Dilks, O. Pechenik, and J. Striker.

This group proved that increasing tableaux under k-promotion and plane partitions—seen as order ideals in $[a] \times [b] \times [c]$—under Striker-Williams promotion are in equivariant bijection. That is, there is a simple bijection from increasing tableaux of shape $a \times b$ with largest entry $a+b+c-1$ and order ideals inside $[a] \times [b] \times [c]$ such that the notion of *promotion* in each setting coincides. O. Pechenik spoke about this result on the last day of the workshop. He concluded his talk with a discussion of the *resonance* phenomenon, both in this setting and in general. The above bijection gives a nice explanation for the observed resonance in increasing tableaux and plane partitions, namely, that the orbit structure of the promotion action indicates there is a projection from these objects under these actions to an underlying object with rotational symmetry.

Problem (6).
Participants: K. Dilks, S. Okada, O. Pechenik, J. Striker, and B. Young.

This group made progress by finding a candidate object with $3n-2$ rotational symmetry to explain observed resonance in alternating sign matrices and certain related plane partitions; it remains to determine the right injection to the underlying objects such that the toggle group action in question reduces to rotation.

Problem (7).
Participants: M. Chan, S. Haddadan, S. Hopkins, and L .Moci.

This group solved (and, after the workshop, significantly sharpened and generalized) a question about Young tableaux with relevance to algebraic geometry. In this problem, natural measures on a-by-b rectangular Young tableaux give rise to measures on lattice paths from $(0,0)$ to (a,b), and one seeks a formula for the expected number of turns in such a path. The group showed that for a wide variety of path-measures of this kind, the expected number of turns is $2ab/(a+b)$, and they have already begun writing up their results for publication.

Problem (8).
The above is not a complete list of the problems that were discussed in the working groups at the workshop. Some of these problems were proposed by participants during problem sessions; see http://web.mit.edu/∼shopkins/docs/aim_dyn_alg_comb_notes.pdfS. Hopkins' notes for the full list of such problems.

Outcomes

The workshop made significant progress in understanding several different discrete dynamical systems, notably by uncovering relationships to existing mathematics. Based on informal feedback, it appears that both junior and senior attendees were excited by the subject area, and found the resulting problems interesting.

Collaborations.
Many of the working groups have continued to discuss their research problems after the workshop. In particular, the groups that worked on Problems (1), (3), (4), (5), and (7) all plan to write joint papers on their results.

Future Impact.
Additionally, J. Propp has created a "dynamical algebraic combinatorics" listserv for continued discussion amongst workshop participants and anyone else who would like to join the conversation.