Entropy power inequalities

organized by
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Workshop Summary

In his classic paper of 1948, Shannon stated the Entropy Power Inequality (EPI), which gives a lower bound on the entropy of the sum of independent continuous random variables, with equality achieved by Gaussian random variables with proportional covariance matrices. Subsequent research has created a variety of new proofs of this result, and analogous results have been proved in a variety of settings. However, there still remain many open questions, which this workshop sought to understand.

We followed the standard 'AIM model', with two expository talks each morning. On Monday afternoon we held a lengthy problem session, which identified around 15 open research problems related to the EPI: following discussion between the organisers, many of these problems formed the basis of group research in the remaining afternoons of the workshop. The groups were formed following a voting procedure recommended by AIM staff. This very smoothly produced the desired result of groups of three to six people working on each problem. It was perhaps unexpected that these groups essentially remained stable for the remainder of the workshop; there was very little recombination of participants, or a move towards other open problems.

We briefly describe the problems that the groups worked on. To summarise overall progress, most of the problems remain open, but some progress was made on each, with most groups intending to continue collaboration to resolve them.

(1) [**Matrix-valued EPIs**] It was remarked that, for one-dimensional Gaussians, the EPI reduces to the additivity of variances. In higher dimensions, it is appealing to think that a corresponding result should exist in terms of matrix-valued quantities, which would reduce to the additivity of covariances in the Gaussian case. This group made some progress in defining potential quantities, through de Bruijn-like integrals, and hope to continue to work on their properties.

(2) [**CLT convergence rates**] The work of Barron and Johnson provided a proof, using elementary tools such as projection operations, of convergence in the Central Limit Theorem at an optimal $O(1/n)$ rate. However, this proof requires the finiteness of the Poincaré constant, which is known to be too restrictive, following the work of Bobkov, Chistyakov and Götze who showed that 4th moment conditions suffice. This group sought to develop the original Barron and Johnson argument, replacing the Poincaré condition with use of the maximal correlation inequality. Some promising results were obtained, but details remain to be resolved.

(3) [**EPI for beamsplitter addition**] The so-called beamsplitter addition operation provides a non-standard way of adding together integer-valued random variables, motivated by ideas in quantum optics. It mirrors addition in Shannon's classical problem, in that the (maximum entropy) geometric family is preserved under this addition operation. Work in the quantum information community has made some progress towards proving conjectured forms of the EPI for this discrete operation. This group focused on developing a purely classical proof of this result, and made some progress in developing a combinatorial understanding of beamsplitter addition.

(4) [**Strengthened Brunn–Minkowski**] Recent work of Courtade has developed a strengthened EPI involving three random variables, at least in the setting where one term is Gaussian. This group worked to prove the natural analogue of this in convex geometry, namely a three-term Brunn–Minkowski inequality where one set is a ball. Some progress was made in understanding special cases, and in gaining insight into whether the Gaussianity (or ball) condition is necessary in general.

(5) [**IID and Gaussianity**] This group considered a conjecture that for IID X, Y with variance 1 and $Z \sim N(0,1)$ the entropy $h(X+Y) \leq h(X+Z)$. By expanding this expression to leading order, a counterexample was obtained, but further possible conditions were proposed under which such a result might hold. A framework was developed to relate this conjecture to monotonicity of entropy on convolution, which gives a strong form of the EPI.

Overall, we feel that the workshop was a success, and would like to thank the AIM staff for their help and support. A key goal was to bring communities (information theory, functional analysis, probability, convex geometry, statistics etc) together, and we were particularly pleased to see that many of the groups exemplified this, with genuine cross-disciplinary links being forged. We intend that this workshop will be the catalyst for future research collaborations and workshops, and will use the AIM problem page to faciliate future collaboration.