

EQUILIBRIUM STATES FOR DYNAMICAL SYSTEMS ARISING FROM GEOMETRY

organized by

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Workshop Summary

This focused workshop studied uniqueness and statistical properties of equilibrium states for dynamical systems with a geometric origin, with an emphasis on geodesic flows and billiard flows. Geodesic flows are an important class of systems, whose study mirrors the historical development of the theory of dynamical systems. Many major theoretical results were obtained first for geodesic flows, before being generalized to broader classes of systems. For compact manifolds of negative curvature, where geodesic flow is uniformly hyperbolic, the theory is by now classical and well-understood, and one can formulate an “ideal theorem” that contains a long list of properties known to be true. Geodesic flows beyond this setting continue to be a crucial motivation and testing ground in the modern theory of dynamical systems, and it remains a substantial task to establish parts of the ideal theorem in the most general setting possible. Directions of contemporary study include weakening the hypotheses on the curvature, weakening the compactness assumption, and weakening the assumption that the underlying space is a Riemannian manifold. A principal goal of the workshop was to communicate the latest techniques in this theory, which have developed rapidly over the last several years, and to explore the many open questions in this area.

The four organizers of the workshop made substantial progress in this direction during an AIM SQuaRE program from 2014–2016, culminating in the paper “K. Burns, V. Climenhaga, T. Fisher, D. J. Thompson, *Unique equilibrium states for geodesic flows in non-positive curvature*, GAFA, Vol. 28 issue 5, 1209-1259 (2018)”, hereinafter “BCFT”. This paper established major parts of the ideal theorem using the “specification property”, which is one of (at least) four distinct approaches to the thermodynamic formalism. In some sense, the workshop was a sequel to our SQuaRE, with the additional goals of building bridges to the other major approaches to thermodynamic formalism, and broadening participation in this research direction by including a new generation of practitioners who have emerged in this area over the last few years.

Two of the talks at the workshop, by Climenhaga and Thompson, described the progress in BCFT, which represents the state of the art in the specification approach. Further talks described other approaches that have seen substantial progress in recent years: Jerome Buzzi and Yuri Lima discussed coding by Markov partitions; Mark Demers discussed transfer operators on anisotropic Banach spaces; and Agnieszka Zelerowicz discussed construction of leaf measures via geometric measure theory.

The workshop also focused on applications of some or all of these four approaches to new settings where the thermodynamic formalism is not yet fully understood. One important setting of this kind is billiard systems, which were included in the talks by Lima and Demers. Another setting is that of translation surfaces, which were described in a talk by Jayadev Athreya; here one can study both the non-uniformly hyperbolic flow on moduli space, as

well as the flow on the translation surface itself. The latter is an important example of a CAT(0) geodesic flow; Dave Constantine gave a talk about general aspects of CAT(-1) and CAT(0) geometry, which generalize many key features of the Riemannian notions of negative and non-positive curvature to more general metric spaces. Finally, talks by Kurt Vinhage and Alena Erchenko discussed rigidity and flexibility results for geodesic flow. There are interesting connections between these results and thermodynamic formalism, and this is an area where recent progress in the theory of equilibrium states could potentially be applied.

The talks described above were chosen to equip the participants for success in the group work part of the workshop. Our problem session on Monday afternoon generated more than 30 questions, and 6 or 7 groups formed for the rest of the week, naturally coalescing around some of the questions raised. We briefly summarize the activities of a selection of the afternoon working groups.

1) **CAT(0) geodesic flows.** A group worked on extending the ideas of BCFT, which apply to non-positive curvature manifolds, to CAT(0) geodesic flow. This is a natural, interesting, and highly non-trivial generalization. The group focused on the important special case of translation surfaces, and made very good progress.

2) **Magnetic flows.** Another team made excellent progress on applying BCFT to the class of magnetic flows. These are a natural generalization of Riemannian geodesic flow, where the presence of a magnetic field defines magnetic geodesics; these are the lowest cost paths of travel in the manifold in the presence of the magnetic field. The main results of BCFT appear to have a natural extension to magnetic flows on surfaces.

3) **Thermodynamics for billiards.** One group brought together experts on billiard systems with experts on the specification and Markov partition approaches. An initial question was whether every positive entropy invariant measure for the Sinai billiard is *adapted*, which is a necessary condition for such measures to be studied via a countable-state Markov partition. The group found a construction of positive entropy measures that are not adapted, which raises the important question of exactly when such measures exist and how much entropy they can carry. The group also studied the possibility of using the specification approach to establish uniqueness of the measure of maximal entropy for Sinai billiard flows. This setting was recently studied by Baladi and Demers using anisotropic Banach spaces, and conversations during the workshop suggest that certain entropy estimates from that work will allow the specification approach to be used to establish uniqueness; the group plans to continue this work in the near future.

4) **Handel-Thurston examples and entropy rigidity.** A group looked at extending Katok's and Foulon's entropy rigidity results beyond the cases in which they are known. Cases of interest include geodesic flow on non-compact negative curvature surfaces, and non-uniformly hyperbolic diffeomorphisms on the torus. The Handel-Thurston example was considered as a potential counter-example to entropy rigidity, but the group checked rigorously that it is not, and moved on to new settings to prove affirmative results.

Other teams worked on random walks and measures at the boundary, variational principles in non-compact settings, and on developing the multifractal analysis in non-compact settings. The opportunity for collaboration provided by the workshop was very valuable. A number of new collaborations were formed, and it is likely that at least three or four of the teams will follow through on the projects initiated at the workshop, and produce high

quality publications in the coming year or so. On the last day, plans were being made to create regular Skype meetings, and for opportunities to meet in the future, have mutual seminar visits, etc. It is notable in terms of broader impact and professional development that many of our early career participants (graduate students, postdocs) were members of groups that made substantial progress, and are now part of new ongoing collaborations and directions of research. Another long-term impact is that the selection of talks gave accessible accounts of all four of the major techniques available in this area, and of some of the most important settings. This has helped clarify future directions of research, and has increased the coherence of the community working in this active area of dynamical systems. In summary, we believe that the workshop was highly productive, and will lead to new work and activity in this research area for years to come.