Equivariant derived algebraic geometry

organized by
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Workshop Summary

1. Summary

This was a very productive week! Following the standard AIM format, we had a series of 6 introductory talks in the mornings Monday through Wednesday. Mike Hill started us off with an introduction to equivariant derived algebraic geometry (eDAG), framing the problems and suggesting broad motivating questions. He also gave a talk on the evidence for eDAG, describing some results from classical derived algebraic geometry and how they assemble to point the way towards eDAG. John Greenlees gave an introduction to equivariant stable homotopy theory, emphasizing the role of isotropy separation and connections with algebraic geometry. Charles Rezk gave an introduction to derived algebraic geometry: a crash course on the necessary tools in the theory of ∞-categories and ∞-topoi and then an overview of the basic results in Lurie's Spectral Algebraic Geometry (née "DAG"). Mark Behrens gave a comprehensive overview of TMF, explaining the basics of the Goerss-Hopkins-Miller sheaf of topological modular forms, the best-understood DAG object to date. Finally Kirsten Wickelgren gave an introduction to motivic homotopy theory, explaining basic terms, proving foundational results, and setting the stage for comparisons with equivariant homotopy.

On Thursday and Friday, we had 5 research talks closely connected to themes in the workshop. Clark Barwick spoke on G-∞-categories, giving a crash course in the ongoing work of his team that provides a quasicategorical framework for understanding equivariant homotopy theory. Lars Hesselholt spoke on periodic topological cyclic homology and the Hasse-Weil zeta function, explaining new progress on Denninger's program to resolve the Riemann hypothesis! Doug Ravenel talked about a new characterization of the slice filtration, describing on-going work which simplifies the slices from the Hill-Hopkins-Ravenel solution to the Kervaire invariant one problem. Charles Rezk spoke about equivariant TMF, explaining some basic results in Lurie's approach together with a new take on complex analytic elliptic cohomology. Finally, Dondi Ellis talked about the homotopy of $MGL\mathbb{R}$, reporting on a motivic generalization of the classical Landweber-Araki analysis of $MU_{\mathbb{R}}$.

There were also two moderated problem sessions: one at the beginning of the workshop to help generate problems for discussions during the week and one at the end to help map the terrain in the field in general. Below we will focus on 3 of the big themes that came up during the workshop.

2. Motivic and Profinite Equivariant Homotopy

For L/k a finite Galois extension, work of Heller-Ormsby produces a functor

$$\mathcal{S}p^{Gal(L/k)} \to \mathcal{S}p_k^{Mot}$$

that lifts the "Galois correspondence"

$$G/H \mapsto \mathrm{Spec}(L^H).$$

Here, the source is the category of $Gal(L/k)$-spectra, and the target is the Morel-Voevodsky category of motivic spectra over k. When $L = \overline{k}$ (a very small class of examples by the Artin-Schreier theorem), then Heller-Ormsby showed that this is a fully faithful embedding (after completion at the Hopf map η).

Since most Galois groups are infinite, this raises the question of how to appropriately make sense of the lefthand side and also to describe the map into the righthand side.

How "rich" is the map $\mathcal{S}p^{Gal(\bar{k}/k)} \to \mathcal{S}p_k^{Mot}$?

Equivariant homotopy theory is only well-developed for compact Lie groups. In particular, there is not a good theory for continuous actions of profinite groups. The biggest obstruction here is working out the right notion of equivariant duality theory and the Wirthmüller isomorphism. What is the right formulation of the Wirthmüller isomorphism for the continuous action of a profinite group? In the classical compact Lie cases, one of the ways to understand G-equivariant homotopy theory is as giving coherent Wirthmüller isomorphisms by work of Blumberg. This records the desired transfer information, and shows that G-spectra can be modeled by G-objects in spectra plus appropriate G-additivity. For finite subquotients of a profinite group, everything then works just as one might expect; one of the ways to understand the issue is to focus on infinite quotients of G. An extremely active area of discussion this week was rebuilding the classical Wirthmüller isomorphism (together with the expected shifting by a representation sphere when $\dim G > 0$) and attempting to formulate the p-adic version.

This project has lots of applications throughout homotopy theory. In addition to the initial question on work of Heller-Ormsby, there are basic questions in classical chromatic homotopy that would benefit from a well-developed profinite equivariant homotopy theory. Work of Devinatz-Hopkins, building on work of Morava, shows that the $K(n)$-local sphere (and more generally, $K(n)$-local finite complexes) is completely determined by the action of the Morava stabilizer group, an n^2-dimensional p-adic Lie group, on the Lubin-Tate spectrum. This can be used to produce surprising duality equivalences between certain $K(n)$-local spectra and their $(-n^2)$-fold suspensions.

3. Diagrammatic approaches to equivariant homotopy

There are many different approaches to describing a "genuine" G-spectrum and constructing the equivariant stable category. The classical approach is to give a space for every finite dimensional representation of G; one then deduces structural properties (e.g., the transfer) by carefully proving results like the Wirthmüller isomorphism and the Adams isomorphism. There is a more recent approach, pioneered by Guillou-May, that focuses on making sense of the diagram of fixed points for a particular G-spectrum as a kind of Mackey functor in spectra, explicitly recording the restriction and transfer maps. This is very close to the intuition for a G-spectrum, but it can still be complicated to work with. (The work of Barwick and his team also proceeds along these lines.)

Work of Greenlees for C_p shows another approach to this kind of problem that has a decidedly algebro-geometric flavor: reconstructing a C_p-spectrum via its Tate diagram. Any

C_p-spectrum E fits into a canonical pullback square in C_p-spectra:

$$E[r][d]\tilde{E}C_p \wedge E[d]F(EC_{p+}, E)[r]\tilde{E}C_p \wedge F(EC_{p+}, E).$$

This has a distinct advantage, since

(1) The C_p-spectrum $F(EC_{p+}, E)$ depends only on the underlying C_p-object in spectra and
(2) $\tilde{E}C_p \wedge E$ and $\tilde{E}C_p \wedge F(EC_{p+}, E)$ are completely determined by their fixed points (the latter of which is the classical Tate spectrum of E.)

Using this, we see that the data of a genuine C_p-spectrum is equivalent to three pieces of data:

(1) an ordinary spectrum E^{gC_p} which plays the role of the geometric fixed points $(\tilde{E}C_p \wedge E)^{C_p}$,
(2) a C_p-object in spectra E, and
(3) a map of spectra $E^{gC_p} \to E^{tC_p}$ from the geometric fixed points to the Tate spectrum.

One of the most widely discussed problems this week was understanding how this decomposition interacts with the C_p-symmetric monoidal structure on the category of C_p-spectra.

How are C_p-commutative monoids represented in the Tate diagrammatic approach to C_p-spectra?

We can also ask the more structural underlying question.

How is the C_p-symmetric monoidal structure on C_p-spectra represented by the Tate diagrammatic approach.

4. Green Algebraic Geometry

One of the biggest stumbling blocks in equivariant derived algebraic geometry is a lack of some of the most basic algebraic geometry tools in the context of equivariant stable homotopy theory. While algebraic geometry has dealt extensively with G-equivariant schemes, constructions in genuine equivariant algebraic geometry must instead handle schemes modeled not on rings with a G-action but rather on Green or Tambara functors.

Basic results about localization of equivariant commutative rings and of Tambara functors show that the notion of a Tambara functor structure will not extend over any of the ways one might immediately think to describe the Zariski site of a Green or Tambara functor. This has substantially impeded progress in this area. However, other classical algebraic notions such as “flat” or “square-zero” can be defined in Green functors exactly as one might hope. In particular, using these observations, the working group easily showed that inverting an element in the value of a Green functor at G/G is a flat operation, just as in the classical case.

The working groups focusing on this had several purely algebraic approaches which seemed quite promising. These produced several checkable questions:

If we define “formally étale” for a map $\underline{R} \to \underline{S}$ of Green functors via lifting over square-zero extensions (just as classically), then is $\underline{S}$ necessarily a flat $\underline{R}$-module? Is the diagonal open if we include a finite presentation hypthothesis? If $\underline{R}$ is a commutative Green functor and we invert an element $b \in \underline{R}(G/H)$ for $H \subsetneq G$, then is $\underline{R}[b^{-1}]$ flat as an $\underline{R}$-algebra?

Thinking about these approaches also resulted in a topological approach via the Tate square:

If we declare that a map of commutative monoids in C_p-spectra $R \to S$ is étale if $\Phi^{C_p} R \to \Phi^{C_p} S$ is étale and the map of underlying commutative monoids $i_e^* R \to i_e^* S$ is étale, then is $R \to S$ flat and is the multiplication map $S \wedge_R S \to S$ a projection onto a Cartesian factor?

These two conditions are one of the equivalent formulations of the notion of étale classically, so this provides a sanity check — which easy computations shows holds. This leads to further questions about whether this notion of étale allows for Goerss-Hopkins-Miller styler arguments about computing spaces of commutative monoid maps.