Equivariant techniques in stable homotopy theory

organized by
Andrew Blumberg, Teena Gerhardt, and Michael Hill

Workshop Summary

Summary

This was a very productive and enjoyable week! In the mornings on Monday, Tuesday, and Wednesday, we had a series of introductory talks aimed at ensuring all of the participants had a solid understanding of the basic questions motivating the workshop. Monday morning, Mike Hill started us off discussing the classical connections between chromatic and equivariant homotopy theory. Agnès Beaudry then described a series of interrelated questions in chromatic homotopy theory, unpacking some of the classical applications of equivariant techniques to the chromatic setting. Tuesday morning, Eva Belmont explored computational techniques in and connections between equivariant and motivic homotopy, and then Lars Hesselholt described some recent advances and computational simplifications in algebraic K-theory made possible by new approaches in trace methods. Wednesday morning, Lennart Meier provided a summary of the interplay between equivariant homotopy theory and spectral algebraic geometry, culminating in a discussion of recent work on equivariant topological modular forms. Gabe Angelini-Knoll then introduced Rognes' "red shift conjecture" in algebraic K-theory, exploring some recent progress in that area.

On Thursday and Friday, we had a four talks that focused more on recent research in the area (although the speakers also worked to provide extensive background and grounding). Thursday morning, Dylan Wilson continued the discussion of red shift, describing his recent work with Hahn to prove this in various fundamental cases. Inbar Klang then provided an introduction to equivariant factorization homology, explaining how various models of equivariant structured ring spectra admit a kind tensoring over G-manifolds. Friday, Vesna Stojanoska returned to the talks from Monday, showing how equivariant homotopy theory techniques can provide a conceptual description of fundamental computations of $K(n)$-local duality phenomena. Finally, John Greenlees showed how to unify various gluing constructions in algebra, equivariant stable homotopy theory, and representation theory.

There was a problem session on Monday afternoon, moderated by Lennart Meier. The problems were collected and written up by Hana Jia Kong and are available on the AIM website. Each afternoon, before splitting into individual working groups, we discussed brief sumamries of the previous days' progress and thoughts for future work.

Below we will focus on 3 of the big themes that came up during the workshop.

Equivariant multiplicative structures

In classical stable homotopy theory, commutativity of a ring spectrum is not a property but rather extra structure: we have explicit homotopies that witness the commutativity of the multiplication and higher coherences. Equivariantly, we can further ask for various homotopies expressing compatibility between these commuting homotopies and G-actions on the tensor powers of the spectrum that also permute the tensor factors. These "norm multiplications" were a key technical ingredient to Hill–Hopkins–Ravenel's solution to the Kervaire invariant one problem. Blumberg–Hill provided a language to describe the possible kinds of norm multiplications on a space or spectrum, introducing the notion of an N_∞-operad. These generalize the classical E_∞-operads by explicitly packaging the additional structure of the norms. Several related questions arose throughout the workshop.

Problem 0.1. *What are the "finite" analogs of the N_∞-operads? In other words, what is an N_k-operad?*

The classical E_k-operads parameterize increasing levels of commutativity (or by Dunn's additivity theorem, families of compatible associative multiplications). By foundational work of May, they also describe k-fold loop space structures. Because of this, we have some operads which we know must be N_k-operads: the little disks for various representations of G. Beyond this, we have very little understanding of the situation. One thing that further impedes progress is a complete lack of analogues of the most basic computations available non-equivariantly since the 1980s.

Problem 0.2. *What are more examples of higher equivariant THH or Γ^G-homology?*

Higher equivariant THH and various incarnations of Γ^G-homology are examples of equivariant factorization homology. Currently, we only have examples for ordinary THH (which takes E_1-algebras) and for THR (a C_2-equivariant theory which takes E_σ-algebras). Beyond this, we have essentially no data.

Spectral algebraic geometry

Spectral algebraic geometry enjoys a close connection with chromatic homotopy theory, often providing natural families of cohomology theories of particular chromatic heights. The functoriality of many of the spectral algebraic geometry constructions also means that groups acting on an underlying variety or stack produce equivariant cohomology theories. Remembering this equivariance often illuminates otherwise opaque constructions and computations.

The best known and studied examples of spectral algebraic geometry objects come from topological modular forms with level structures. In this case, groups acting on the level structures produce equivariant spectra. For example, elliptic curves together with a point of exact order n (corresponding to a spectrum $Tmf_1(n)$) admits an action of $\mathbb{Z}/n^\times$ which moves the points of exact order n around, and hence when n is odd, we can view these as C_2-equivariant objects. Meier has shown that the Hill–Hopkins–Ravenel slice filtration of these C_2-spectra is particularly simple, and this has allowed a host of related questions about the Picard group and duality to be easily solved.

Problem 0.3. *Given a C_2-spectrum arising from algebraic geometry, how do we calculate the slices?*

One obstruction to making conjectures for this problem is the lack of worked examples beyond the theory of topological modular forms and a few examples of topological automorphic forms. We now have a host of interesting cohomology theories with good properties, but we know very little about even their coefficient rings.

Problem 0.4. *What are the homotopy groups of examples of theories of topological automorphic forms or equivariant topological modular forms?*

More geometrically, we can ask about what vector bundles are orientable for various algebraic geometry theories. For example, Spin bundles are orientable for real K-theory, and String bundles are orientable for topological modular forms. For general theories arising from algebraic geometry constructions, we have no way to predict what bundles will be oriented. Equivariant homotopy theory provides a way to rewrite this question.

Problem 0.5. *Given a collection of homotopy elements $S \subset \pi_\star\big(MU^{(G)}\big)$, how can we predict what representations are orientable in $MU^{(G)}[S^{-1}]$?*

Finally, we can ask what the equivariant structure can tell us about algebraic geometry questions. Work of Hausmann gives a universal property for the homotopy groups of equivariant bordism, generalizing Quillen's seminal work. This theory is the prototype of an equivariant commutative ring spectrum, so it has all norms. We do not know what these tell us, geometrically.

Problem 0.6. *What is the Tambara functor structure on the homotopy of MU_G and what information does it record?*

Odd primary Realness

At the prime 2, recent work of Hahn–Shi and of Hill–Hopkins–Ravenel have allowed equivariant techniques to directly access computations in chromatic homotopy theory. As shown by Beaudry–Hill–Shi–Zeng, this allows us to turn many previously inaccessible computations into more understandable and conceptual ones using the Fujii–Landweber Real bordism $MU_{\mathbb{R}}$. These constructions all rely on the existence of an index 2 subfield of $\mathbb{C}$, and hence do not obviously generalize. A fundamental question here is to build the bordism theory that plays the role of $MU_{\mathbb{R}}$ at odd primes.

Problem 0.7. *Can we construct a highly structured C_p spectrum BP_{μ_p} with the properties*

1. *the underlying spectrum is $BP^{\wedge(p-1)}$ and*
2. *the geometric fixed points are $H\mathbb{F}_p$?*

These two features are a distillation of the actually used pieces of $MU_{\mathbb{R}}$; conjectures of Hill–Hopkins–Ravenel show that a spectrum like BP_{μ_p} with these properties should have predictable slices and a slice spectral sequence like that of $MU_{\mathbb{R}}$ and its norms. In fact, this could be used to settle the fate of the final unknown chromatic height 2-family: $\beta_{3^i/3^i}$ at the prime 3 (the analogous elements for primes bigger than 3 were shown not to exist by Ravenel and for $p = 2$, these are the Kervaire classes).

Work of Behrens–Shah building C_2-equivariant homotopy theory out of motivic homotopy theory over $\mathbb{R}$ provides some promising avenues for attacking this problem, as it rewrites equivariant stable homotopy as a kind of localization (as Greenlees described in his talk on Friday).

Problem 0.8. *What is the odd primary analog of motivic homotopy over $\mathbb{R}$?*

Part of understanding this focused on understanding more basic structural questions about motivic stable homotopy and the connections to equivariant homotopy via Heller–Ormsby's work embedding $Gal(K/k)$-equivariant stable homotopy into motivic homotopy over a ground field k. In particular, some time was spent unpacking the Burklund–Hahn–Senger notion of an Artin–Tate motive.