Exact crossing numbers

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Workshop Summary

Two main problems were the focus of this workshop.

(1) The Harary-Hill Conjecture (Anthony Hill, 1958), which states that the crossing number $cr(K_n)$ of the complete graph K_n is

$$H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

(2) The Zarankiewicz Conjecture (Paul Turán, 1944), which states that the crossing number $cr(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ is

$$Z(m,n) := \left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor.$$

Overview

The Harary-Hill Conjecture has only been confirmed for $n \leq 12$. The best known lower bound $0.8594 \cdot H(n) \leq cr(K_n)$ was proved by De Klerk, Pasechnik, Schrijver in 2005. The upper bound $cr(K_n) \leq H(n)$ is given by Hill's constructions from 1958, which are a particular instance of cylindrical drawings. Only one more infinite family of drawings was known to achieve exactly $H(n)$ crossings. This family, found by Blažek and Koman in 1962, is a particular instance of 2-page book drawings. Very recently, Ábrego, Aichholzer, Fernández-Merchant, Ramos, and Salazar proved the conjecture for a large class of drawings called *shellable*. These drawings include cylindrical and 2-page book drawings.

Regarding Hill's Conjecture, two main questions were considered during the workshop. First, is it possible to extend the proof by Ábrego et al. to a larger class of drawings? And second, are there other infinite families of drawings that achieve $H(n)$ crossings?

Zarankiewicz's Conjecture has only been confirmed when $\min\{m,n\} \leq 6$ (Kleitman, 1971), when $m = 7$ and $n \leq 10$, and for the special cases $(m,n) = (8,8)$ and $(m,n) = (8,10)$ (Woodall, 1993). In 2006, DeKlerk, Pasechnik, and Schrijver proved the lower $0.85 \cdot Z(m,n) \leq cr(K_{m,n})$. Very recently, Norine used flag algebras to improve this lower bound to $0.91 \cdot Z(m,n) \leq cr(K_{m,n})$ (Norine explained this result in a talk during this workshop). One variant of this problem is to only consider *rectilinear* drawings, that is, drawings where the edges are straight line segments. Since rectilinear drawings achieving $Z(m,n)$ crossings are known, it is natural to conjecture the same identity in the Zarankiewicz Conjecture for the rectilinear case. Essentially the same results are currently known for the topological and the rectilinear cases.

During the workshop, the rectilinear case was considered for a reduced class of drawings called line separated. A rectilinear drawing of $K_{m,n}$ is *line separated* if the two classes of vertices determining the bipartition are separated by a straight line. Line separated drawings of $K_{m,n}$ with exactly $Z(m,n)$ crossings are known, so in order to gain insight on Zarankiewicz's Conjecture it makes sense to focus in such drawings.

Outcomes

Several results were achieved during the workshop and collaborations are expected to continue. Here is a list of main results and future plans.
Bishellability. The notion of shellability was generalized to *bishellability*. For a non-negative integer s, a drawing D of K_n is *s-bishellable* if there exist sequences $a_0, a_1, \ldots, a_s$ and $b_s, b_{s-1}, \ldots, b_1, b_0$, each sequence consisting of distinct vertices of K_n, so that, with respect to a reference face F the following two conditions are met: (1) for each $i = 0, 1, 2, \ldots, s$, a_i is incident with the face of $D - \{a_0, a_1, \ldots, a_{i-1}\}$ that contains F and b_i is incident with the face of $D - \{b_0, b_1, \ldots, b_{i-1}\}$ that contains F; and (2) for each $i = 0, 1, \ldots, s$, $\{a_0, a_1, \ldots a_i\} \cap \{b_{s-i}, b_{s-i-1}, \ldots, b_0\}$. The Harary-Hill conjecture was proved for $(\lfloor n/2 \rfloor - 2)$-bishellable drawings. There are drawings that are $(\lfloor n/2 \rfloor - 2)$-bishellable and not $(\geq \lfloor n/2 \rfloor)$-shellable. Such a drawing with 11 vertices has been recently found. This drawing is crossing optimal. Understanding bishellable drawings is part of the future plans in this topic.
New families. A new infinite family of drawings of K_n that achieves $H(n)$ crossings was found. This family is neither shellable nor bishellable. In fact, every edge is crossed by at least another edge. A second family was found using sector matrices. This family satisfies that the drawing of K_{2m+1} contains the worst possible drawing of K_{2m} implied by the Harary-Hill conjecture. Currently, these new families of drawings are been studied looking for some insight on what is needed to prove the Harary-Hill conjecture for drawings that are neither shellable nor bishellable.
Line separated drawings of $K_{m,n}$. Flag algebras were used to improved the previously best known bound on $Z(n,m)$ in the rectilinear case for a reduced class of drawings. Namely, drawings were the two classes of vertices that determine the bipartite graph are separated by a straight line. This crossing number is denoted by $cr(K_{n|m})$. When $m = n$, the bound was improved to $cr(K_{n,n}) \geq 0.99Z(n,n) + O(n)$. There is hope that a refinement of the methods used during the workshop will lead to the proof of (the asymptotic version of) the conjecture in this specific case, that is, $cr(K_{n,n}) = Z(n,n) + O(n)$. The case when n and m are not necessarily equal is currently being considered.
k-planar crossing number. The k-planar crossing number $cr_k(G)$ of a graph G is defined as the minimum number of crossings in a drawing of G where the edges are partitioned into k subsets, and a crossing is counted if and only if the edges crossing are in the same subset. Define $\alpha_k = \inf\{c \geq 0 : cr_k c(G) \leq c \cdot cr(G) \text{ for all } G\}$. It was known that $1/36 \leq \alpha_2 \leq 3/8$. The bounds on α_k were improved to

$$\frac{1}{k^2} \leq \alpha_k \leq \frac{2k-1}{k^3}.$$

As a corollary, α_k approaches 0 when k approaches ∞. There was no improvement for the 3/8 bound on α_2. Currently, the problem where cycles of size 4 are not present in G is being considered.

Other problems. We list now other problems that were considered during the workshop and are currently being studied by workshop participants.

(1) Finding a way to define k-edges for topological drawings of the complete graph that are not necessarily good drawings and finding an identity (or at least an inequality) that relates k-edges to the odd-crossing number and the pair-crossing number of the complete graph.
(2) Understanding how the set of k-edges changes when the same drawing on K_n on the sphere is projected onto the plane is different ways.
(3) Improving the current bounds on the maximum number of edges that a tangled thrackle can have.
(4) Understanding the structure of crossing optimal rectilinear drawings of the complete graph. It has been conjectured that for each n there is a crossing optimal rectilinear drawing of K_n with triangular symmetry and whose first $n/9$ convex layers are triangles.
(5) Let $K_r(rn)$ be the complete r-partite graph on rn vertices $K_{n,n,\ldots,n}$. Conjecture an explicit value $A(r,n)$ for the crossing number of $K_r(rn)$, so $A(1,n) = H(n)$ and $A(2,n) = Z(n,n)$. Then find the limit of $A(r,n)/CR(K_r(rn))$ as $n \to \infty$, where $CR(K_r(rn))$ is the maximum number of crossings in a drawing of $K_r(rn)$.

Closing Remarks

Based on the outcomes listed above, the workshop proved to be a success. There were active groups working on a diversity of problems, with tangible new results. There has been post-workshop activity in all of these problems. Moreover, to our knowledge, there are writeups of at least two of these new results, one of which has been already submitted.

There is another less tangible but potentially equally important highlight of this workshop: new collaborations started, some of them involving researchers that had had essentially no contact with this part of Combinatorics. Explicit examples include Sergey Norine (senior researcher), Florian Pfender (mid-career), and Bernard Lidicky (junior researcher). This workshop exposed these colleagues to an area outside their field of expertise, arising their interest and actively engaging them in new, hopefully lasting and long reaching, collaborations with researchers well-seasoned in the main themes of this workshop.