isher-Hartwig asymptotics, Szego expansions, and applications to statistical physi

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Workshop Summary

Introduction

The workshop gathered experts from roughly two related fields, namely Fisher–Hartwig asymptotics for Toeplitz determinants and semi-classical expansions of traces of Wiener–Hopf operators. Both have important applications to problems in statistical physics, random matrix theory and spectral theory. The gathering at AIM was a unique opportunity to discuss the recent advances and future directions.

Below we outline the points addressed in the morning lectures and afternoon working group discussions.

Entanglement entropy (EE)

In the physics literature there has been a sustained interest in entanglement entropy (EE) over the last fifteen years or so. A large number of relevant results was based on approximate arguments and numerical work. In 2005, Gioev and Klich noticed the connection between the spatial scaling of the entropy for the free Fermi gas in its ground state and an (open at that time) conjecture of Harold Widom in the asymptotic analysis of Wiener–Hopf operators, dating back to 1982. This conjecture deals with a multi-dimensional, continuous generalization of a two-term asymptotic Szegő-type expansion of Toeplitz determinants. In two opening lectures two of the organizers (Spitzer and Sobolev) explained this connection and the mathematical progress which has been reached in recent years. Despite this progress many basic but challenging questions remain unanswered even for such simple models as the free Fermi gas. We explain some of them below; our http://www.fernuni-hagen.de/angewandte-stochastik/download/statement.pdfstatement can be used for further details.

Double scaling limits.

Recently, the organizers proved the spatial scaling asymptotics of the EE for the free Fermi gas in an equilibrium state at positive temperature $T > 0$. In some cases also the double scaling limit, when both scaling parameter L and the temperature $T \leq T_0$ vary independently, was studied. Both for constant T and for $T \downarrow 0$, the leading asymptotic coefficient is quite complicated. If T is fixed, then the EE is of order L^{d-1} and is expressed as a multi-fold integral, even for spatial dimension $d = 1$. By completely different, non-rigorous methods, a few years ago Korepin derived a much simpler (so-called universal) expression (for $d = 1$) different from ours. One working group discussed possible cases when both formulas give the same answer. One possibility is the regime when $TL = const.$ as $L \to \infty$.

Complete asymptotic expansion.

Once the leading asymptotic expansion has been proved the obvious question is about next-to-leading terms. For smooth bounded symbols f and smooth bounded functions g, Widom has long proved that there is a complete asymptotic expansion [1]

$$\operatorname{tr} g(\mathbb{1}_{L\Omega}(Q) f(\Delta) \mathbb{1}_{L\Omega}(Q)) = a(g,\Omega)L^d + b(g,\partial\Omega)L^{d-1} + c(g,\partial\Omega)L^{d-2} + \cdots .$$

Is there still a complete asymptotic expansion if g is non-smooth, such as e.g. $g(t) = |t|^\alpha$ for $\alpha > 0$ or the binary entropy function $g(t) = -t\ln(t) - (1-t)\ln(1-t)$ for $t \in [0,1]$? It is not trivial to show that for such non-smooth functions $b(g,\partial\Omega)$ is well-defined. Is the coefficient $c(g,\partial\Omega)$ finite as well and does it display the correct asymptotics?

In the one-dimensional setting of Toeplitz determinants, Basor reported on a three-term asymptotic expansion. We discussed extensions to the leading asymptotic expansion for the EE and the result by Jin and Korepin. Korepin himself presented his results on the (discrete) one-dimensional spin-1/2 XY model, which includes the term of order one. Such a result is not yet proved in the continuous case of free fermions on the real line $\mathbb{R}$, not to mention the higher dimensional case.

Non-smooth boundary.

The asymptotic formulas were derived under some smoothness conditions on the spatial domain Ω and the Fermi surface. For heat-kernel expansions it is well-known that corners produce additional terms. If Δ_Γ is the Dirichlet Laplacian on a polygonal domain $\Gamma \subset \mathbb{R}^2$, then the term of order one in tr $e^{t\Delta_\Gamma}$ as $t \downarrow 0$ contains a simple function of the angles of the polygon.

At which order do we see contributions from corners in the asymptotics of the EE? This question has been actively discussed but no definite conclusion has yet been reached as the problem seems to be very hard.

External fields.

In his talk, Stolz considered the EE of the one-dimensional spin-1/2 XY chain in a random magnetic field transversal to the XY interaction. Contrary to the case without such a field, the scaling of the EE is now asymptotically constant. Then he also presented a new result on the related many-body localization in the one-dimensional, anisotropic, ferromagnetic spin-1/2 Heisenberg model in a random magnetic field.

Some afternoon discussions touched upon the generalization to the free Fermi gas in an external (scalar) potential V. In particular, when studying the EE, we are led to the analysis of the traces $\operatorname{tr} g(\mathbb{1}_{L\Omega}(Q) f(\Delta + V(Q)) \mathbb{1}_{L\Omega}(Q))$ as $L \to \infty$. The difference with the similar trace without V is called impurity entanglement. Suppose V is a short-range potential, that is, $V \in \mathrm{L}^1(\mathbb{R}^d)$. At which order of the asymptotic expansion of the above trace do we see the impact of a local perturbation?

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[1] Q is the position operator, in other words, the basic multiplication operator on $\mathrm{L}^2(\mathbb{R}^d)$, $\mathbb{1}_\Omega$ is the indicator function of the bounded, measurable set $\Omega \subset \mathbb{R}^d$, and Δ is the Laplace operator on $\mathbb{R}^d$.

Delta-Fermi gas

Tracy and Widom reported on their recent progress on the ground-state energy of the attractive delta-Fermi gas. Here, fermions on the real line interact pairwise via a delta-potential of strength $2c \leq 0$. Let $e_0(\rho, c)$ denote the ground-state energy per particle with $\rho > 0$ denoting the uniform mean particle density. Then they proved an expansion of $e_0(\rho, c)$ for small c of the form

$$e_0(\rho, c)/\rho^2 = \frac{\pi^2}{12} + \frac{c}{2\rho} + O((c/\rho \ln(\rho/|c|))^2)\,.$$

The starting point of their analysis is the non-linear Gaudin equation and a Wiener–Hopf analysis of a corresponding Toeplitz operator.

Several extensions of this result have been discussed at the workshop. For instance, one may look at the ground-state energy of the delta-Fermi gas in a subspace with non-vanishing total spin, that is, the number of fermions with up-spin is (macroscopically) different from the number of down-spins. The single Gaudin equation is then replaced by two coupled non-linear integral equations. The analysis of Tracy and Widom then leads to a 2×2 matrix Wiener–Hopf problem but it is not immediately clear how to proceed from there.

Another interesting generalization occurs when we look at a discrete version of the delta-Fermi gas. This is in fact the one-dimensional Hubbard model. We are not aware of rigorous results for an expansion of the ground state energy for small coupling constants.

EE in the delta-Fermi gas is another interesting question but, from a rigorous point of view, at the moment it seems to be out of reach because it goes beyond the reduced one-particle description.

Fisher–Hartwig type transition asymptotics for Toeplitz determinants

Presently, this topic occupies a central place in the study of Toeplitz determinants. Therefore, it received a special attention at the workshop.

So let us consider Toeplitz matrices

$$T_n(f_t) = (f_{t,j-k})_{j,k=0}^{n-1}\,, \qquad f_{t,j} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, f_t(e^{i\theta})e^{-ij\theta}\,,$$

where the complex-valued symbol f_t depends on a parameter t. The symbol f_t is of standard form

$$f_t(z) = e^{V(z)} z^{\beta_1+\beta_2} \prod_{j=1}^{2} |z - z_j|^{2\alpha_j}\, g_{z_j,\beta_j}(z)\, z_j^{-\beta_j}$$

with two Fisher–Hartwig singularities at the points $z_1 = e^{it}$ and $z_2 = e^{i(2\pi-t)}$. The parameters α_1 (at z_1) and α_2 (at z_2) describe power- or root-type singularities, whereas β_1 and β_2 describe jump discontinuities. Furthermore,

$$g_{z_j,\beta_j}(z) = \begin{cases} e^{i\pi\beta_j} & \text{if } 0 \leq \arg(z) < \arg(z_j) \\ e^{-i\pi\beta_j} & \text{if } \arg(z_j) \leq \arg(z) < 2\pi \end{cases}.$$

Krasovsky (in joint work with Claeys) reported on the asymptotic behavior for the determinant of T_n which is uniform in t within a certain range that depends on n. Their result settles a conjecture of Dyson on Bose–Einstein condensation in the one-dimensional delta-Bose gas.

We discussed even more general situations of emerging and merging singularities whose symbol depends on n and possible applications.

Simm presented results on random unitary matrices. He considered an integral of a Toeplitz determinant with merging Fisher–Hartwig singularities as above. In a certain regime the results of Claeys and Krasovsky apply but in other regimes they lead to yet unproven conjectures of Fyodorov and Keating (2012).

Claeys talked about his recent work with another participant of the workshop (Charlier) on thinning and conditioning of the circular unitary ensemble. The various probabilistic quantities can be expressed in terms of Toeplitz determinants and orthogonal polynomials on the unit circle, and Claeys–Charlier use these expressions to obtain asymptotics for large matrices.

Ising model

McCoy gave a presentation on the two-dimensional Ising model. It was discussed how to obtain non-linear differential equations of Painlevé type for spin correlation functions and for Fredholm determinants by means of isomonodromic deformation theory. This requires extensions of theorems of Geronimo and Case and of Borodin and Okounkov. Another question concerns the susceptibility of the Ising model and natural boundaries.

The last talk was delivered by Bleher on the 6-vertex model and the relation to random matrices.

Final remarks

We are convinced that some of the ideas that were discussed at this workshop will lead to new results and that new collaborations will be started. We enjoyed excellent talks and vivid discussions about ambitious problems.