Finite tensor categories: their cohomology and geometry

organized by
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Workshop Summary

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The goal of the workshop was to make progress on a long standing "finite generation of cohomology" conjecture for finite dimensional Hopf algebras and more generally finite tensor categories. A second objective was to develop applications to the subject of tensor triangular geometry in the context of representation theory of those finite tensor categories for which finite generation is known. The idea was to bring together mathematicians currently working on these questions from several different angles, facilitating the development of new approaches and calculations, establishing new collaborations, and leading to new discoveries.

The first four mornings there were two introductory talks, with one on Friday morning. They provided insights on the topics developed in this workshop as well as information on recent results on the direction of finite generation of the cohomology in different contexts. The speakers were:

Monday:
- Sarah Witherspoon, *Cohomological finite generation conjecture*,
- Paul Balmer, *Introduction to tt-geometry.*

Tuesday:
- Juan Omar Gómez Rodríguez, *Cohomological Finite Generation for Finite Group Schemes*,
- Milen Yakimov, *Non-commutative tt-geometry.*

Wednesday:
- Guillermo Sanmarco, *Introduction to the Finite Tensor Categories*,
- Istvan Heckenberger, *Nichols algebras.*

Thursday:
- Stanislas Herscovich, *Cohomology of the Fomin-Kirillov algebra*,
- Cris Negron, *Finite generation of the drinfeld double of a finite group scheme.*

Friday:
- Vera Serganova, *Tt-geometry for supergroups.*

Problem list and initial progress

On Monday afternoon, there was a problem session moderated by Kent Vashaw, at which around twenty open problems were proposed. Over the week, the conference participants worked on the following six problems.

Finite Generation of Cohomology for Bosonizations.

Participants. Nicolás Andruskiewitsch, David Jaklitsch, Van Nguyen, Amrei Oswald, Julia Plavnik, Anne Shepler, Harshit Yadav, Xingting Wang.

Project Description.

Nicolás Andruskiewitsch posed the following question on the conjecture on the finite generation of cohomology (FGC for short):

> Consider an exact sequence of finite-dimensional Hopf algebras over a field
>
> $$\longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow$$
>
> If A and B satisfy the FGC, does C as well? In other words, which extensions of finite-dimensional Hopf algebras preserve the FGC property?

We started by considering the case of abelian extensions, i.e., extensions with A commutative and B cocommutative, following ideas of [Andruskiewitsch and Natale 2024]. We then considered general conditions on extensions that would allow us to use ideas of [Negron 2021] on equivariant deformations and ideas of [Negron and Pevtsova 2023]. We focused on arguments that would apply especially to the case when C is a smash product algebra and considered the more general case of crossed products. We identified certain desired conditions on extensions and outlined an argument that may give the FGC for these extensions.

Tame Hopf algebras and Balmer spectrum.

Participants. Karin Erdmann, Julia Pevtsova, Vera Serganova, Sarah Witherspoon.

Project Description.

Discussions began with the concrete goal of finding the Balmer spectrum of a specific example, namely modules for the small quantum super group $u_q(\mathfrak{sl}(1|1))$ for a primitive nth root of unity q. The answer was found on the first day to be the projective line modulo the action of the cyclic group C_n. Then the team considered other Hopf algebras of tame representation type (except characteristic 2). After thinking about known examples, the team conjectured that the Balmer spectrum for each tame finite dimensional Hopf algebra is homeomorphic to a projective line. Due to known classification results for such Hopf algebras, this looks possible to prove.

Rank varieties for quantum complete intersections.

Participants. Karin Erdmann, Cris Negron, Julia Pevtsova, Vera Serganova, Sarah Witherspoon.

Project Description.

The team of previous project worked on a second question, motivated by the frequent utility of rank varieties in proving the tensor product property for support varieties. Let be a field and let $\Lambda_{\mathbf{q}} = \langle x_1, \ldots, x_m \rangle / (x_i x_j - q_{ij} x_j x_i,\ x_i^{n_i})$, a quantum complete intersection associated with a matrix $\mathbf{q} = (q_{ij})$ and positive integers $n_i \geq 2$. Assume these parameters arise from a braiding on a vector space of dimension m via action of a finite group G and also that $\Lambda_{\mathbf{q}}$ is a Nichols algebra. Let $A_{\mathbf{q}} = \Lambda_{\mathbf{q}} \# G$ be its bosonization. We want to define a rank variety that will be homeomorphic to the cohomological support variety. The homogeneous case (all n_i equal and all q_{ij} equal for $i < j$) is known. The nonhomogeneous case is unknown and not immediately obvious, while at the same time appearing naturally in some quantum group and pointed Hopf algebra settings. The team discussed first a rank 2 example arising from $u_q(\mathfrak{sl}(1|2))$, in which case $A_{\mathbf{q}}$ can be viewed alternatively as a semidirect product $R\#G$ for a commutative algebra $R = [s,t]/(s^N, t^2)$. In this case we explored several possible definitions of rank variety: 2

[itemsep=0pt, leftmargin=*,label=(0)]π-points, noncommutative hypersurfaces, commutative hypersurfaces, matrix factorizations.

The first two ideas do not seem to work. We spent the most time discussing the last two ideas in combination, that is, converting the problem to a commutative one, and leveraging known commutative hypersurface support theory.

The team is optimistic about both projects, and plans to continue discussing remotely.

Cohomology of FK.
(4) *and smaller interesting Nichols algebras over nonabelian groups of group-likes*

Participants. István Heckenberger, Stanislas Herscovich, Héctor Peña Pollestri (also Milen Yakimov on the first day).

Project Description.

We discussed some of the techniques for computing projective resolutions of quadratic algebras, with special focus on the Fomin-Kirillov algebras FK(3) and FK(4) (see [MR1667680]). Even though we know what the Yoneda algebra of FK(3) is (it is just a polynomial extension $\mathrm{FK}(3)^![\omega]$ of the quadratic dual $\mathrm{FK}(3)^!$, see [MR3459024,MR4102553]), we do not understand where this element ω really comes from.

Since FK(4) was too complicated (see [arXiv:2309.02611]), we moved to other finite dimensional Nichols algebras. More particularly, after considering possible geometric tools that could be helpful to simplify computations, we discussed the extra algebraic structure of the quadratic dual $A^!$ of quadratic algebras A obtained as the quadratic part of some of the known examples of finite dimensional Nichols algebras in characteristic zero (see http://mate.dm.uba.ar/~lvendram/zoo/http://mate.dm.uba.ar/~lvendram/zoo/ and [MR2803792,MR3356939]). In particular, we noted the role in that algebraic structure of $A^!$ played by the groups and racks that are naturally used in the definition of the Nichols algebras: $A^!$ can be regarded essentially as a twisted monoid bialgebra.

We decided to do more computations in detail in order to better understand what structures precisely appear and the rôle they play in the determination of the corresponding Yoneda algebra. In particular, using GAP the cohomology of the quadratic part A of the second example in http://mate.dm.uba.ar/~lvendram/zoo/http://mate.dm.uba.ar/~lvendram/zoo/

seems to satisfy $\mathrm{Ext}^\bullet_A(,) = A^![\omega]$ with ω in cohomological degree 4 and internal degree -6, precisely as in the case for FK(3).

The cohomology of group schemes in the Verlinde category.

Participants. Iván Angiono, Agustina Czenky, Alexei Davydov, Eric Friedlander, Pablo Ocal, Guillermo Sanmarco, Antoine Touzé.

Project Description.

We dealt with the following problem, suggested by Vera Serganova:

(1) Let p be a prime, let Ver_p be the Verlinde category, and let $X \in \mathrm{Ver}_p$.
  (1) Compute the cohomology of the group scheme $GL(X)$.
  (2) Compute the cohomology of some finite group schemes in Ver_p.

The group considered the finite generation of cohomology of group schemes in the Verlinde category Ver_p. The goal was to prove this property for the group scheme $GL(X)$ where $X \in \mathrm{Ver}_p$, and use this as a springboard to prove the finite generation conjecture for finite group schemes in Ver_p.

The group began studying the cohomology of the group $GL(X)$. Its complexity compelled the group to switch to studying the restricted Lie algebra $u(\mathfrak{gl}(X))$, which had been a successful strategy in the classical case. The group observed that in some cases, $u(\mathfrak{g}) \cong U(\mathfrak{g})$ because of certain vanishing occurring in positive characteristic, and given it appeared simpler, it switched to understanding $U(\mathfrak{g})$. For this, a categorical Chevalley–Eilenberg complex $C^{CE}_\bullet(\mathfrak{g})$ was constructed. This complex is well defined in any symmetric monoidal category, and its terms are projective $U(\mathfrak{g})$-modules. When specialized to $-\mathsf{Vec} \simeq \mathrm{Ver}_2$ and $-\mathsf{sVec} \simeq \mathrm{Ver}_3$, this complex recovers the usual Chevalley–Eilenberg resolution and the May resolution, respectively. However, $C^{CE}_\bullet(\mathfrak{g})$ is not a resolution in general. The group proved that $C^{CE}_\bullet(\mathfrak{g})$ is a (projective) resolution of the monoidal identity in Ver_p if and only if the underlying object of $\mathfrak{g}$ in Ver_p has only L_1 and L_{p-1} factors. Moreover, the group computed $\mathrm{H}^\bullet(C^{CE}_\bullet(\mathfrak{g}))$ explicitly. The main tools for this were a spectral sequence with vanishing differentials, the existence of a PBW basis of $\mathfrak{g}$, and the Künneth formula.

The group plans on continuing working on this project. It already has some progress on modifying $C^{CE}_\bullet(\mathfrak{g})$ to obtain an actual resolution, as well as ideas to understand the structure of the symmetric algebra of the simple objects $S(L_m)$ for $1 \leq m \leq p-1$.

Noetherianity of the Balmer spectrum of stable and derived categories.

Participants. Paul Balmer, Karthik Ganapathy, Juan Omar Gómez Rodríguez, Cris Negron, Kent Vashaw, Milen Yakimov.

Project Description.

Let $\mathbf{C}$ be a finite symmetric tensor category. Our goal was to consider the relationship between (1) cohomological support satisfying the tensor product property, (2) the finite generation conjecture of Etingof–Ostrik holding for $\mathbf{C}$, and (3) the Noetherianity of the Balmer spectrum of the stable category $\underline{\mathbf{C}}$ or the derived category $D^b(\mathbf{C})$. A very ambitious form of this question amounts to asking whether the comparison map

$$\mathrm{Spc} D^b(\mathbf{C}) \to \mathrm{Spec}^{\mathrm{h}}\, \mathrm{H}^\bullet(\mathbf{C})$$

is a homeomorphism. By tensor-triangular localization techniques, it is possible to reduce to the local case, that is, to a situation where we wish to prove that the comparison map

$$\mathrm{Spc}\mathbf{K} \to \mathrm{Spec}^{\mathrm{h}}\,\mathrm{H}^{\bullet}(\mathbf{K})$$

for some tensor-triangulated category $\mathbf{K}$ which has a Noetherian, local, periodic cohomology ring satisfies the property that the fiber of the irrelevant ideal of $\mathrm{H}^{\bullet}(\mathbf{K})$ is a single point. We proved this fact for $D^b(\mathbf{C})$ by a direct analysis of the perfect complexes. While we know that the comparison map cannot be a homeomorphism in the broadest generality (since the stable homotopy category provides a counterexample), we hope that there exists a describable class of tensor-triangulated category, closed under localization, for which this is true. As a special case, we asked: if $f : \mathbf{1} \to X$ and $g : \mathbf{1} \to Y$ are two morphisms such that $f \otimes g = 0$, then is it true that either f or g is nilpotent on some nonzero object? This is true if X or Y is invertible, since this would imply that f (resp. g) is nilpotent on its own cone. We attempted to generalize this proof to arbitrary morphisms in $D^b(\mathbf{C})$.