Flag algebras and extremal combinatorics

organized by
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Workshop Summary

The workshop Flag Algebras and Extremal Combinatorics was hosted by American Institute of Mathematics from Oct 13 to Oct 17, 2025. The organizers were József Balogh (UIUC), Dingding Dong (Harvard), Bernard Lidický (Iowa State University), Annie Raymond (UMass Amherst).

Scientific program

As a fairly young research field, combinatorics was seen by some mathematicians in other fields as a discipline of ad-hoc methods, with very little theory. This has changed in the last few decades with the ascent of deeper theories and methods, e.g., the probabilistic method and the regularity method. These theories and methods have been driving a large part of modern developments in combinatorics.

In this workshop, *flag algebras* was the main technique explored. This method, developed by Razborov in 2007, allows one to solve combinatorial problems via streamlined calculations that combine elements from optimization and computer engineering. It led to many recent breakthroughs on long-standing open problems of Erdős, Sós, Turán, Gromov and Zarankiewicz, to name a few.

The method of flag algebras is typically computer-assisted and sometimes relies on computational power. Computational power has increased significantly since the first wave of results following the discovery of flag algebras. One of the goals of the workshop was to consider certain problems that previously could not be solved because of computational power, but could perphaps now be solved.

Moreover, recent work has combined tools from algebraic geometry (sums of squares theory and tropicalization) to better understand the strengths and limitations of flag algebras, e.g., to prove inequalities that generate all valid inequalities of a certain form and to find classes of problems that cannot be solved with the method. Another goal of the workshop was to keep studying problems along these lines.

The workshop attracted experts in the computational aspects of flag algebras as well as researchers exploring the theoretical limitations of sum of squares methods.

The workshop followed the typical schedule for AIM workshops with talks in the morning and problem solving in the afternoon. The workshop started with talks that introduced the method on induced graph densities and explored theoretical limits using homomorphism densities. Later talks focused on areas of applications of the method and improvements in understanding the method.

Talks

The workshop had 26 participants. It included 6 graduate students and 6 postdoctoral scholars.

Ten participants, including two postdoctoral scholars, delivered talks. There are nine recorded sessions available at https://vimeo.com/showcase/flagextremal. The talks were delivered by the following speakers.

(1) Grigoriy Blekherman (Georgia Tech),
Graph profiles
(2) Daniel Brosch (University of Klagenfurt), (postdoctoral scholar),
Derivatives in flag algerbas
(3) Dingding Dong (Harvard), (postdoctoral scholar),
Solving a Turán problem for partially directed hypergraphs using flag algebra
(4) Bernard Lidický (Iowa State University),
Introduction to flag algebras
(5) Dhruv Mubayi (University of Illinois at Chicago),
Inducibility in Graphs
(6) Florian Pfender (University of Colorado Denver),
Progress on Ramsey numbers
(7) Annie Raymond (University of Massachusetts, Amherst),
Flags flying in a different direction
(8) Christoph Spiegel (Technische Universität Berlin),
Solving large problems
(9) Jan Volec (Czech Technical University in Prague),
Tricks for flag algebras
(10) Fan Wei (Duke University)
Domination exponents for cycles

Collaborative Activities

The participants explored several different problems posed during the workshop. Several of the groups are still meeting. The typical problem of maximizing or minimizing the number of induced copies of a particular graph was extended to explore this quantity by also fixing the number density of edges. These problems are called density profile problems. It seems that we do not currently have efficient tools to approach these problems in general. The focus of several of the groups is to develop such tools. Below is a summary of progress of some of the groups.

$K_4 + I_4$.
During the workshop, Dhruv Mubayi posed the question of maximizing the number of induced copies of K_4 plus its complement in graphs of fixed edge density. During the workshop, a partial progress was made to resolve the conjecture. Following the workshop, the conjecture was fully resolved by a flag algebra calculation that is not computer-assisted. The group is currently working on extending the result to $K_k + I_k$ for all $k \geq 5$.

3-Profiles of Graphs.
During the workshop, Annie Raymond suggested a problem of finding the graph profile of

all four graphs on 3 vertices, i.e., the closure of all points of the form (d_0, d_1, d_2, d_3) where d_i is the induced density of the graph on 3 vertices with i edges. Since densities of all four 3-vertex graphs sum to one, the result is a 3-dimensional object. The group has numerical evidence and guesses for some parts of the boundary of the object. The group has met several times and improved the guess of the shape over the meetings and has started to prove certain parts.

$\{K_2, C_5$.

-profile During the workshop, postdoctoral scholar Felix Clemen suggested to find the maximum density of an induced C_5 given some edge density. The group had a good guess on how what the $\{K_2, C_5\}$-profile should look like including some numerical evidence. However, the guess turned out to be incorrect as the group later discovered that what seemed to be a numerical anomaly was actually an indication of a better construction. The problem also became more approachable when restricted to triangle-free graphs. In this case the group has identified construction for almost the entire profile and is working on turning numerical solutions into exact arithmetic.

Log homomorphism density.

During the workshop, we introduced the notion of log density for a weighted graph or graphon G. We are interested in understanding which graphs or graphons have log density 1. Following the workshop, Aldo Kiem provided a short proof showing that for 2-step graphons with parameters $W = ((a, b), (b, c))$ satisfying $b \leq \max(a, c)$, the log density is equal to 1. He also constructed counterexamples showing that this statement fails in general when $b > \max(a, c)$. The group has continued to meet regularly to better understand this phenomenon and is working toward a full classification of 2-step graphons for which the log density is equal to 1.

Cuts in graphs.
We have discussed two problems on balanced bipartitions of K_4-free graphs. We have shared insights on our failed attempts to solve these problems so far (both with flags and without). The most interesting ideas revolved around using flags with colored vertices and edges to obtain stronger cuts for uncolored graphs and possible ways of using flags to prove bounds on expressions including the maximum function for some very specific graph settings. Some remarks made during the workshop led to progress for these problems.

Permutation shattering.
We are studying the maximum number of shattered 3-point subset of $[n]$ using a six-tuple of permutations of order n. Since the conjecture on this maximum, if true, is coupled with a tight bound, it remains to find a matching construction. We are currently thinking of some more computer-driven (heurisitic) search, and simultaneously we are inspecting various 3-uniform hypergraph constructions that have the correct density with a hope they might be relevant to the hypergraph reformulation of the shattering question.

Uniform Turán density.
We have looked at the recent palette result of Ander Lamaison on determining the uniform Turán densities in 3-uniform hypergraphs via auxiliary extremal problems in edge-colored graphs. At the moment, it seems to us one cannot directly use flag algebras in a general palette setting and rather one needs to "distillate" an ad-hoc extremal question tailored to

a particular hypergraph, say the tetrahedron, one has in their mind. So-far, it is not clear what this auxiliary problem should be in the case of tetrahedron.

Schur triples.
We discussed the problem of minimizing the number of monochromatic Schur triples in 3-colorings of $[n]$ as well as some other related problems. We used Fourier techniques to approach the problem and study the properties of optimal colorings. Some partial progress was made optimizing various terms with computers. There was also some discussion of leveraging known results on the minimal number of monochromatic triangles in 3-colored graphs to get a bound in the arithmetic setting.

Explicit fractalizer.
During the workshop, Aldo Kiem and Fan Wei fully resolved the problem of finding an explicit non-trivial fractalizer, i.e., for any n, the unique example maximizing induced copies is the iterated balanced blow-up of itself.