Rigidity and flexibility of microstructures

organized by
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Workshop Summary

Overview of the Area.
Combinatorial and Geometric rigidity theory is concerned with the local and global uniqueness of congruence classes of *frameworks* as solutions to their underlying geometric constraint system. Intuitively, the framework is *locally rigid* if it cannot be continuously deformed in the ambient space while satisfying the constraints, and it is *globally rigid* if there is no other framework that is a solution or realization of the underlying constraint system. Frameworks are categorized by the types of geometric primitives and constraints. Example categories include bar-and-joint (point and distance), panel-and-hinge (e.g. polyhedra), and point-line-angle. When the constraints are inequalities, examples include tensegrities, as well as packings of discs and spheres. Such constraint systems are ubiquitous in nature (aperiodic and periodic structures in proteins, crystals, colloids and other materials), in 3D printing (microstructure and metamaterials design), and elsewhere.

The study of geometric rigidity can be traced back to historical figures such as A. Cauchy who, in 1813, proved convex polyhedra are rigid. Subsequent work extending Cauchy's theorem by Dehn, Alexandrov and Pogolerov continued into the 40's and 50's. Similarly, combinatorial rigidity can be traced to J. C. Maxwell who, in 1864, developed *necessary* combinatorial, graph-based, conditions for bar-joint structures to be rigid. Hilda Polaczek-Geiringer proved in 1920 that Maxwell's condition is sufficient in 2 dimensions.

For most categories of frameworks, each constraint typically involves a pair of primitives. Hence the framework and the constraint system have an underlying constraint graph of vertices representing the primitives and edges representing the constraints. Generic properties of the constraint system or the framework are often purely combinatorial properties of the underlying constraint graph. The useful interplay between the geometric and the combinatorial was further apparent in e.g. the relation between planar framework self-stresses, Maxwell-Cremona reciprocal diagrams, and polyhedral liftings; and the relation between conformal maps and Koebe's theorem from the 30's on rigidity of circle packings corresponding to maximal planar contact graphs.

Talks.

The workshop began with Robert Connelly describing links between rigidity of frameworks and circle packings in 2-dimensions. In particular he reported on joint work with Steven Gortler where they have obtained a new proof of the celebrated Koebe-Andreev-Thurston theorem using techniques from bar-joint rigidity theory.

Next Jozsef Solymosi described the famous unit distance problem of Erdos: what is the maximum number of unit distances among n points in 2-dimensions. He explained the connection between this problem and the problem of finding certain point-line incidence structures, and how the rigidity of unit distance bar-joint frameworks in the plane may provide a new approach to this notoriously difficult problem.

On the Tuesday Brigitte Servatius explained work of Walter Whiteley on k-plane matroids and how they can be used to characterise when an incidence graph has a realisation as an independent identified body-joint framework in 2-dimensions. This generalizes the generic bar-joint rigidity results in dimension 2 to a large class of nongeneric cases.

The second Tuesday talk was shared between Herman Servatius and Bernd Schulze. Herman provided a detailed background on local, infinitesimal and static rigidity before Bernd explained some recent work on periodic frameworks: infinite graphs which are realised in a way that preserves periodicity with respect to a given fixed lattice.

The Wednesday talks began with Shin-ichi Tanigawa giving the Henry Crapo memorial talk of the workshop. He presented new research with Katie Clinch and Bill Jackson where they have explained how to analyze the C_2^1-cofactor matroid and the maximal abstract 3-rigidity based on matroid erections. Interestingly, the theory of matroid erections was first introduced by Henry Crapo in the 1970's.

Next Orit Raz gave a detailed presentation of a theorem of Lovasz which suggests an approach to provide a polynomial time algorithm for computing the rank of the rigidity matroid in dimension 3. Lovasz considers the dimension of $span(F \cap h)$, where F is a family of linear subspaces of $\mathbb{R}^d$, h is a subspace of codimension 1 in $\mathbb{R}^d$, and $F \cap h := \{f \cap h \mid f \in F\}$. He gives a formula for this dimension akin to that of the Dilworth truncation of a submodular function, expressed as a minimum over all partitions of F of the sum of the dimensions of the individual part intersections with h, given that h satisfies a certain "general position" assumption. The result Orit presented was an explicit way to find the minimizing partition, leading to an alternative strongly polytime algorithm for computing the said dimension by submodular optimization (the previous one used Dilworth truncation of submodular functions).

On Thursday Miranda Holmes-Cerfon spoke about a new notion arising from colloidal cluster microstructure; specifically her recent work with Steven Gortler and Louis Theran on a new concept they have termed almost rigidity of bar-joint frameworks. For a given framework, they provide two numbers $\eta_1 < \eta_2$ where $\eta_1 = 0$ implies rigidity and $\eta_2 = \infty$ implies global rigidity (neither converse holds). Almost rigidity is defined by η_1 being small in relation to η_2. The flexing neighborhood defined by η_1 is small compared to the larger neighborhood where other equivalent frameworks can be obtained by flexes involving intermediate edge length change by η_2.

Ciprian Borcea then talked about periodic frameworks with respect to a fully flexible lattice. He gave a historical perspective of the notion of a d-periodic graph, and emphasized the need for a community to adopt historically consistent terminology to avoid confusion. He recapped several results about rigidity of higher dimensional periodic bar-joint and body-hinge frameworks, and a formalization of deformation of periodic frameworks, giving examples of auxetic or expansive deformations along the way.

On Friday, Meera Sitharam presented an array of results, questions and ideas from her group, relating to previous talks, and sketched applications for microstructures. These included a result on uniqueness of a minimal flip path between certain triangulations, a definition of a flip in dimension 3 and a result on rigidity of a flip-equivalent space of corner-sharing tetrahedral structures, results on finding so-called flex-paths between (equivalent) isostatic frameworks obtained by removing one edge at a time, flexing the 1-dof mechanism, and replacing an(the) edge; and efficient algorithms for finding all frameworks of a given planar Laman graph and edge lengths, as well as applications to crack formation, tunnelling in 2D glassy networks, sticky sphere packings, and spaces of microstructures for efficiently designing and 3D printing objects with arbitrary specified macro shape and macro physical properties.

Problems discussed in afternoon break out sessions.

Point-line incidences and unit distance rigidity (Mozhgan Mirzaei, Tony Nixon, Orit Raz, Brigitte Servatius, Meera Sitharam, Jozsef Solymosi and Louis Theran). The group decided to work on the following problem. Suppose you have n points in 2-dimensions, cn^2 collinear triples and no collinear 4-tuples. Show that a large sub-structure of this point line arrangement is affinely rigid with this special geometry. Progress was made both by using regularity to obtain a better understanding of the local structure of the hypergraph of the arrangement and towards understanding affine rigidity without a generic hypothesis.

Construction of d-periodic graphs (Ciprian Borcea, Sean Dewar, John Hewetson, Miranda Holmes-Cerfon, Orit Raz, William Sims and Ileana Streinu). This group worked on understanding which periodic graphs could be obtained from a finite simple graph by an operation which identifies d pairs of non-adjacent vertices.

Global rigidity for triangulations of closed surfaces (Katie Clinch, Jim Cruickshank, Tony Nixon and Shin-ichi Tanigawa). Fogelsanger proved that the graph of a triangulation of any closed surface is infinitesimally rigid in 3-dimensions, and this group discussed extending this to say that a triangulation of any closed, non-spherical, surface is globally rigid in 3-dimensions if and only if the triangulation is 4-connected. There seem to be two interesting approaches to this. One is to extend work of Jordan and Tanigawa on nondegenerate stresses and alongside that obtain new topological contraction operations which reduce the genus of the surface. The second is to use an idea of Bill Jackson which reduces the problem of showing global rigidity is preserved by the vertex splitting operation to an infinitesimal rigidity problem with coincident points, for the application at hand this approach would require an analysis of Fogelsanger's proof where a pair of coincident points is allowed.

Existence of inversive distance packings (John Bowers, Katie Clinch, Robert Connelly, Herman Servatius). Let T be a triangulation of the unit sphere with real weights w_{ij} on the edges. When is it possible to produce a a mapping (realization) of vertices to circles such that the inversive distance between circles i and j is w_{ij} for all edges ij in the triangulation? The problem is solved for certain intervals of inversive distance values; but is well known and difficult for the remaining cases. The group made some progress in understanding the obstacles via examples.

Sticky disk and sphere packings (Remi Avohou, John Bowers, Jim Cruickshank, Rahul Prabhu, Herman Servatius, Meera Sitharam, Louis Theran and Jeremy Youngquist). Let G be planar and Laman. Does there exist a packing of disks with generic radii such that G is the contact graph of this packing? Moreover in 3-dimensions, under what conditions does the neighborhood of a packing in the semi-algebraic set have dimension $4n-m$ where n is the number of vertices and m is the number of edges? Does the packing support an edge-length equilibrium stress (no in dimension 2)? Can 2-banana and nucleation-free flexible circuits be realized with generic radii? This group made some progress on understanding these hard problems, for example by showing that every 2-tree (a subclass of planar Laman graphs) can be realised as a generic radii packing.

Periodic frameworks and spiderweb stresses (Sean Dewar, Bernd Schulze and Shin-ichi Tanigawa). Suppose that $(\tilde{G}, \tilde{p})$ is periodic in $\mathbb{R}^2$ (with respect to a fixed lattice L), such that there exists an equilibrium stress which is strictly positive on all edges and $(\tilde{G}, \tilde{p})$ is infinitesimally rigid. Is it true that $\tilde{G}$ is globally rigid in $\mathbb{R}^2$ for any L-generic realization? The group found that this result is true and essentially known in the crystallography community due to Sunada. They were able to obtain their own proof during the week using rigidity theoretic techniques.

Unit distance body-joint collinear frameworks (Remi Avohou, Orit Raz, Brigitte Servatius, Louis Theran). This group considered the following problem. Is it possible to strengthen a theorem of Walter Whiteley on body-pin frameworks in $d = 2$, to say that $G = (B, J, I)$ has an independent realization in $\mathbb{R}^2$ where each body has all its joints at unit distance from each other $\iff$ G satisfies $2i \leq 3b + 2j - 3$ (plus subgraph counts), where i is the number of incidences, j is the number of joints, and b is the number of bodies? The group made some progress in understanding motions that arise in this nongeneric setting and in connecting this problem to the first problem.

Independence of 5-regular graphs in 3-dimensions (Tony Nixon. Brigitte Servatius, Herman Servatius, Meera Sitharam). Given a k-regular graph G, is it true that G is independent in the generic 3-dimensional rigidity matroid if and only if G is $(3, 6)$-sparse? A result of Jackson and Jordan reduces the question to the case when $k = 5$. The idea from the group is to use the recent understanding of the maximal abstract 3-rigidity matroid obtained in papers of Sitharam and Vince and Clinch, Jackson and Tanigawa - and possibly a geometric interpretation of the cofactor matroid C_2^1 based on Bézier polynomials that yields panels and hinges in special position - to obtain sufficient understanding of flexible circuits to resolve the question. Note also that the analogous question is easily checked to have a positive answer in 2-dimensions and a negative answer in 4-dimensions.

Many other problems were posed, but were not worked on due to time constraints. A sketch follows. More details can be found at the AIM problem list page.

Measurement set and rigidity of graphs (Posed by Louis Theran) Suppose G is redundantly rigid in $\mathbb{R}^d$. If H is another graph with n vertices and m edges with the same measurement sets over complex coordinates, where the measurement is just the dot product of the edge vector with itself without conjugates, i.e. $M_G = M_H$, then is G isomorphic to H?

Disk Packing on the Unit Sphere (Posed by Robert Connelly) Given a packing of disks on the unit sphere each with radius $\leq \epsilon$ such that the packing has a nontrivial infinitesimal flex, is the packing unjammed for some ϵ?

Maximum Contacts in Unit Disk Packing (Posed by Miranda Holmes-Cerfon) Find a collection of n objects (e.g. disks of given radii) such that all packings have k contacts where k is at most the number required for isostaticity for that type of object but there exists a rigid packing that has a non-trivial infinitesimal flex and an equilibrium stress.

Rigidity of periodic contact graphs (Posed by Bernd Schulze) Let $G = (V, E)$ be a multigraph with $|V| > 1$ and let $(\tilde{G}, \Gamma)$ be the contact graph of a periodic packing in $\mathbb{R}^2$ with group Γ, with Γ-generic radii. Let (G, ψ) be the Γ-labelled quotient gain graph. Is it true that (G, ψ) is $(2, 3, 2)$-gain-sparse? Moreover if $|E| = 2|V| - 2$, is it true that $\tilde{G}$ is rigid?

Improving Lovasz' theorem related to Dilworth truncation (Posed by Orit Raz) In L. Lovasz's paper [”Flats in Matroids and geometric graphs”], he considers the dimension of $span(F \cap h)$, where F is a family of linear subspaces of R^d, h is a subspace of codimension 1 in $\mathbb{R}^d$, and $F \cap h := \{f \cap h \mid f \in F\}$. He proves that this dimension can be expressed in terms of elements of F, given that h satisfies a certain “general position” assumption. The question is: (i) Can the general position assumption in Lovasz's theorem be relaxed? and (ii) Can one find a formula (in whatever terms) that applies to subspaces h that are not necessarily in general position?

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