Floer theory of symmetric products and Hilbert schemes

organized by
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Workshop Summary

Floer theory for symmetric products and Hilbert schemes

Overview

This workshop was inspired by recent developments in the study of Lagrangian Floer theory of symmetric products of Riemann surfaces and Hilbert schemes of symplectic 4-manifolds, and their applications to both symplectic and low-dimensional topology. Some motivating goals were:

- Finding a rigorous mathematical foundation for Aganagic's proposed formulations of Khovanov and knot Floer homology in terms of holomorphic curves in the symmetric product of a surface, and connecting those formulations or her proposed representation-theoretic construction of Khovanov homology to other symplectic constructions (due to Cautis-Kamnitzer or Seidel-Smith), or to ordinary Heegaard Floer homology.
- Leveraging recent developments of Polterovich-Shelukhin and Cristofaro-Gardiner-Humiliére-Mak-Seyfaddini-Smith in symplectic topology arising from Floer theory in symmetric products of Riemann surfaces to address additional questions in symplectic dynamics and build connections between Heegaard Floer theory and symplectic topology. In particular, applying computational methods developed in Heegaard Floer theory to shed light on new problems in symplectic topology.

Thus, the workshop aimed to unite experts in physics, symplectic topology and dynamics, and Floer theory in low-dimensional topology, in hopes that collaboration between these groups would lead to fruitful progress.

Talks

The workshop featured the following talks by participants:

Day	Speaker	Topic
Dec. 5	Mina Aganagic	Link invariants from mirror symmetry, I
Dec. 5	Sobhan Seyfaddini	Hamiltonian homeomorphism groups, I
Dec. 6	Mina Aganagic	Link invariants from mirror symmetry, II
Dec. 6	Ko Honda	Symplectic constructions of Hecke algebras
Dec. 7	Dan Cristofaro-Gardiner	Hamiltonian homeomorphism groups, II
Dec. 7	Ben Webster	B-side models for knot invariants
Dec. 8	Alexei Oblomkov	Categorification of $\mathfrak{gl}(m\|n)$ invariants
Dec. 8	Cheuk Yu Mak	Symplectic geometry of Hilbert schemes
Dec. 9	Peng Zhou	Coulomb branches and superpotentials
Dec. 9	Egor Shelukhin	More problems on Hamiltonian groups

Working groups

Problems in Hamiltonian dynamics.

Based on the recent advances in surface dynamics, a group tried to address a long standing question about the Hofer geometry of surfaces, namely, whether there is a sequence of Hamiltonian diffeomorphisms whose Hofer distance to the set of autonomous Hamiltonians goes to infinity. This question has been resolved for all surfaces other than the sphere; the spherical case is notoriously difficult. The group came up with a possible candidate of such sequence but was not (yet) able to verify the distance tends to infinity. There were also further discussion of the higher dimensional cases, and some promising ideas towards answering this question for the 4-torus.

Another question for higher dimensional manifolds this group discussed was the Lagrangian Poincaré recurrence problem, and the possibility of finding new non-displaceable Lagrangian links to address this question. Two classes of Lagrangian links were tried. One class seemed promising but less interesting. The other class seemed more interesting but it was less clear if one can prove its non-displaceability using Floer theory, which is essential in order to study its Lagrangian Poincaré recurrence.

Resolutions of the diagonal.

Given a symplectomorphism $\phi\colon (M,\omega) \to (M,\omega)$, the graph Γ_ϕ of ϕ is a Lagrangian submanifold of $(M \times M, \omega \oplus (-\omega))$. Under suitable technical conditions, the Lagrangian intersection Floer homology of Γ_ϕ and the diagonal $\Delta = \Gamma_{Id}$ is isomorphic to the fixed-point Floer homology of ϕ. One can view Γ_ϕ as a bimodule over the Fukaya category $F(M,\omega)$. Under further assumptions, the Hochschild cohomology of $F(M,\omega)$ with coefficients in Γ_ϕ agrees with the Floer homology $HF(\phi) = HF(\Delta, \Gamma_\phi)$: both are formulations of the complex of bimodule maps from the diagonal bimodule to Γ_ϕ. A similar phenomenon occurs for other Lagrangian correspondences from (M,ω) to itself.

Similarly, in Khovanov homology, there are various constructions of bimodules associated to braids or tangles. Hochschild homology with coefficients in those bimodules gives various forms of closures, either in a solid torus or in $S^2 \times S^1$. These constructions have led to new invariants and new structures on existing ones.

In both cases, to compute the Hochschild complexes, it is helpful to have an efficient resolution of the diagonal bimodule Δ, in terms of some convenient set of generators. Aganagic's proposed symplectic formulation of Khovanov homology gives several particular generating sets of Lagrangian branes. A group worked to find explicit resolutions of the diagonal using Aganagic's branes and to use those for computations. While the group was unable to find finite resolutions (which may not exist), it proposed an explicit infinite resolution. At present, it is unclear if that resolution will be useful for computations. A longer-term goal of this project is as a tool for extending Khovanov homology to knots in other manifolds, perhaps by considering knots which are braided with respect to an open book decomposition, which might admit Hochschild homology-type descriptions.

B-side analogues of Heegaard Floer theory.

A long-standing question in the general area of mirror symmetry is finding an algebro-geometric construction of Heegaard Floer homology. One group considered this question in the context of Aganagic's invariants. In the simplest case of 2-braid links, they were able to reduce the proposal to known results about homological mirror symmetry for curves. The case of braids of higher rank was pursued but did not lead to concrete results, because the definitions are not completely settled. A related question concerning the relationship between Oblomkov-Rozansky's approach to knot-invariants of Lie supergroups and Aganagic's proposal was also considered.

Khovanov homology for knots in general 3-manifolds.

There has been a lot of interest in extending Khovanov homology from links in $\mathbb{R}^3$ to links in other manifolds. There are currently several proposals for doing so. Extensions to some specific manifolds—thickened surfaces and $S^2 \times S^1$—have been developed and are reasonably well understood. There is a construction by Morrison-Walker-Wedrich extending Khovanov homology to arbitrary 3-manifolds and 4-manifolds, inspired by skein algebras; while the proposal is mathematically precise and there has been meaningful progress on understanding its properties, many basic questions about it are currently unanswered. There are also conjectural extensions, including a proposal of Witten's, whose mathematical foundations still need substantial work.

As mentioned above, Aganagic proposed a Floer-theoretic construction of Khovanov homology that looks in some ways simpler than the existing symplectic Khovanov homology. A group at the workshop explored ideas for using this proposal as a basis for invariants in other 3-manifolds. The group reports interesting discussions but, so far, no concrete developments.

Conjectural new constructions of knot Floer homology.

Based on physical intuition, Aganagic sketches a Floer-theoretic construction of a knot homology associated to any miniscule representation of a quantum group or super-group. In the case of $\mathfrak{gl}(1|1)$, her lab has done computations for this invariant, which is expected to agree with Heegaard Fleor homology for knots. At the workshop, a group worked to substantiate this proposal by proving invariance for this Floer-theoretic invariant of knots directly, without passing through either physics or a resolution in terms of generators of a Fukaya category. The first interpretation of the proposal, which reduced to counting curves in a symmetric product of a bridge diagram not passing over any of the ends of the bridges, was invariant under certain moves, but seemed not to be invariant under a kind of handleslide needed to relate different bridge diagrams. A second proposal, with an incidence condition for curves passing over the endpoints of bridges, seemed not to give a chain complex. Aganagic then gave a tentative revision to the second proposal.

Khovanov homology via Hilbert schemes of $\mathbb{C}$.
$^* \times \mathbb{C}$

This workshop led to an opportunity to revive a proposal of Abouzaid and Smith for constructing a symplectic model of Khovanov homology as a Floer homology groups of non-compact Lagrangians in $\mathbb{C}^* \times \mathbb{C}$. On the second day, one of the groups verified that this proposal is consistent with Aganagic's in the case of braids on 4-strands. A group with a different composition then attempted to compute this invariant for the 2-component unlink later in the week. The computation agreed with the expected answer, and turned out to be easier than expected. We expect that this group will continue their investigation of this program.