Frames for the finite world: Sampling, coding and quantization

organized by
Sinan Gunturk, Goetz Pfander, Holger Rauhut, and Ozgur Yilmaz

Workshop Summary

Our workshop addressed application-motivated problems in finite dimensional frame theory. The attendees represented a wide spectrum of research areas, ranging from pure and applied mathematics such as Lie theory, functional analysis, approximation theory and harmonic analysis to electrical engineering, such as coding theory and signal processing. The experts were chosen to address a small number of central problems in finite frame theory; for example, rate-distortion theory for frame quantization (including quantization for compressed sensing) and construction of deterministic measurement matrices. The workshop was organized so that each of the first three days focused on a particular topic; these were "concrete frames" for Monday, "frame theory and compressed sensing" for Tuesday, and "quantization" for Wednesday. Lead by the discussions during this three day period, we focused on applications of compressed sensing, particularly to analog-to-digital (A/D) conversion, and a discussion of open problems in the area in the remaining two days. Below, we summarize the discussions that ensued during the workshop in chronological order.

Day 1. The first day of our workshop, dedicated to *concrete frames*, was aimed at addressing questions at the very core of frame theory. Pete Casazza started the day with an overview talk, and posed a number of important problems. The most prominent role in Casazza's talk was given to the recently introduced concept of fusion frames and its open problems. Other problems highlighted by Casazza included the "Paulsen problem" and certain frame partitioning problems that relate to the Kadison-Singer conjecture.

The first hour of the afternoon session was devoted to the discussion of the state of the art frame theory proper as well as fusion frames among the full group. In addition to this, we introduced the participants to the structure and the goals of AIM style workshops. For the remainder of the afternoon we split into groups to address the following problems:

- RIP for "admissible" coefficient sets: Often in applications, e.g., when working with redundant wavelet expansions, the full frame at hand lacks useful properties such as the restricted isometry property (RIP), yet one would like to determine sparsest possible representations. The question is to formulate properties that are useful for this purpose, e.g., when it is known that the family of frame elements with nonzero coefficients is a well conditioned Riesz bases.
- Paulsen problem: This problem asks whether one can always find equal-norm Parseval frames that are "close" to a given "almost equal-norm" and "almost Parseval" frame. Each of these two properties (equal-norm and Parseval) can be achieved by a specific operation on any given frame, suggesting an alternating iterative procedure. However this turns out to be a naive idea as these two operations tend to undo each other's effects. The group investigated the obstacles for the lowest dimensional example.

- Fusion frames: In traditional frame theory, one considers frame operators which are sums of rank one projection operators; in fusion frame theory, these operators are sums of higher rank orthogonal projection operators. The group investigated various real-world and mathematical applications of fusion frames as well as discussion of the mathematical problems.
- Special classes of finite frames, e.g., Weyl-Heisenberg frames.

Day 2. The second day of our workshop was opened by a lecture from Ron DeVore on sparse representations and compressed sensing. His two-part talk laid out the theoretical foundations for compressed sensing and alluded to predecessor theories such as Gelfand widths in high dimensional geometry. In the afternoon, reports from the previous day were followed by a discussion session on compressed sensing moderated by Ingrid Daubechies. A number of important concrete and conceptual problems were formulated during this session:

- Dealing with a given matrix: What does compressed sensing tell those of us who have to live with a given matrix that does not satisfy RIP, e.g., a matrix that is associated to measurements from existing sensors, radar, geophysics in a real-world case?
- Motivation for constructing RIP matrices: Why are we building RIP matrices? Examples include the need to build better "sensing devices" for "sparse objects" (e.g., to get away with fewer measurements in MRI/medical imaging), to reduce dimensionality for problems posed (somewhat artificially) in very high dimensions. RIP is the only noise-robust way that we know at this point to achieve these goals.
- Worst case scenarios: Do we have to beat the devil in our detection, i.e., defeat the worst possible case? How would a particular application (prior on the signals) influence our design of the sensing matrices? Is this problem interesting mathematically?
- Probabilistic analysis: There are currently two types of probabilistic estimates, one involving randomness in the matrix and the other involving the distribution or support structure of sparse vectors. What is the role of probability in a deterministic setting?
- Unknown sensing (measurement) matrix: What if we don't know the sensing matrix? How would we find it?
- Probabilistic RIP: Suppose we know that with high probability (over the subsets T) that $\phi_T^*\phi_T$ has its eigenvalues in $[1-\delta, 1+\delta]$. Can we obtain recovery guarantees in this case?
- Substitutes for RIP: Is there a better (e.g. invariant) way to formulate RIP? Is the null-space property (NSP) "necessary and sufficient"?
- Deterministic constructions of measurement matrices: What is a sensible definition of a deterministic construction? (Simply asking for a finite algorithm is not enough since, e.g. ± 1 RIP matrices can always be found in finite time. Do we mean polynomial time?)

In the afternoon, a sizable group addressed the question whether any measurement matrix with the null space property is equivalent to a matrix with RIP in some canonical way. This group continued working on this problem until the end of the workshop and made progress in refining the question. Another group studied a recent deterministic construction of measurement matrices developed by Calderbank et al.

Day 3. The main theme of Wednesday was quantization. We had three morning talks, focusing on different aspects of quantization in the setting of frames. In the first talk of the day, Sinan Güntürk summarized the mathematical theory of coarse quantization, outlining recent results on the accuracy of sigma-delta quantization. Özgür Yılmaz continued on this theme, however his talk focused on the case of finite frame quantization. The third talk was by Vivek Goyal, who discussed the rate-distortion theory of compressed sensing. The afternoon session started with a discussion moderated by Yang Wang. Some of the problems and topics that were formulated during this session were:

- A/D conversion with compressed sensing,
- Robustness of compressed sensing against errors in the measurement matrix,
- Noise shaping for compressed sensing (can we quantize compressed sensing measurements in a manner that is tailored for, e.g., ℓ^1 reconstruction?),
- Modifying sigma-delta quantization methods to adopt them to given frames, and possibly to fusion frames,
- How can we come up with a "quiet" sigma-delta quantization scheme, i.e., one that would produce zero output for prolonged periods of very small input.

The discussion groups working on the problems of noise shaping for compressed sensing, quiet sigma-delta, and adopting sigma-delta to fusion frames made progress during the remainder of the workshop.

Day 4. During many of our discussions during the first three days of our workshop, arguments were exchanged regarding the impact of ideas in finite frame theory, in particular of compressed sensing, to developments such as the Rice one-pixel camera, as well as to quantization methods for compressed sensing measurements. The prominent role of Petros Boufounos, Thomas Strohmer, and Joel Tropp in these discussions motivated us to ask each of them to give a brief impromptu talk on Thursday to share their views on the applications ahead in finite frame theory. After a short discussion and reports from the previous days, we formed afternoon discussion groups. One example arising as a consequence of the talks addressed the well-known Peak-to-Average Power Ratio (PAPR) of trigonometric polynomials used in digital communication. Here, a study group quickly developed a substantial idea for cost-sensitive PAPR reduction; the validity of this idea has now been shown through numerical experiments. In parallel, several groups that were formed during the previous days continued their work.

Day 5. On the final day of our workshop, Ingrid Daubechies summarized the key problems in finite frame theory that were addressed during the workshop. She isolated three settings in which the measurement matrices arise: as a frame determined by the class of objects under study, as a construction (by us) under constraints, and as given by the physics of the problem at hand. Pertaining to the first setting, she pointed out the need for compressed sensing results that allow for the recovery of sparse coefficient vectors for some (special), but not necessarily all support sets. She also emphasized the need for frame-specific quantization schemes. Her remarks were followed by reports on results achieved during the week.

The after-lunch session contained a wrap-up discussion, and successively, an opportunity for groups formed during the week to continue their work. A surprise "AIM song" (or should we call it the "FrAIM song"?) sung to the tune of "New York, New York" with lyrics[1] appropriated to the theme of the workshop was the closing event which we had fun having.

Like almost all participants at our workshop, this was our first experience with an "AIM style" workshop. We were very positively surprised to see that the participants enthusiastically adopted the format. We had a very productive and enjoyable time in Palo Alto – thanks AIM!

---

[1]Such as: *We would be lost and feeling lame without a frame, but find an end to this shame, right here at AIM!*