Frame theory intersects geometry

organized by

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Workshop Summary

Overview. This workshop focused on open problems in frame theory whose formulation contains a significant component from algebraic geometry. Bringing researchers in frame theory and algebraic geometry together spawned significant progress and generated additional topics of interest.

The two main topics of the workshop were the construction of real and complex equiangular tight frames and phase retrieval, that is, the reconstruction of a vector, up to an overall unimodular factor, from the magnitudes of frame coefficients.

The first topic, the construction of equiangular tight frames has a long history. Frames are families of vectors $\{f_j\}_{j\in J}$ in a real or complex Hilbert space $\mathcal{H}$ which allow stable expansions. This is usually phrased in terms of frame bounds $A, B > 0$,

$$A\|x\|^2 \leq \sum_{j\in J} \|\langle x, f_j\rangle|^2 \leq B\|x\|^2 \text{ for all } x \in \mathcal{H}\,.$$

Equiangular tight frames are finite families that combine tightness with specific geometric properties, meaning we can choose equal frame bounds $A = B$ and there is $b > 0, c \geq 0$ such that for all $\{j, l\} \subset \{1, 2, \ldots, N\}$

$$\|f_j\| = b\,, \text{ and } |\langle f_j, f_l\rangle| = c\,.$$

Traditionally, group representations and graph-theoretic constructions were the main source of equiangular tight frames, starting with Seidel's work and recently continued in the complex setting. The workshop promoted an alternative perspective, the connection with algebraic geometry by reformulating the desired propertiess in terms of quadratic equations, splitting them into real and imaginary parts, and attempting to prove or disprove the existence of real solutions for these complexified equations. Even in low dimensions, this task is already very challenging and lead to the computer-aided technique based on Gröbner bases.

The second main topic of the workshop concerned phase retrieval. A main question in this context is whether there are frames $\{f_j\}_{j=1}^N$ for $\mathbb{F}^M$, with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, such that the map

$$A : \mathbb{F}^M/\mathbb{T} \to \mathbb{R}^N, A([x])_j = |\langle x, f_j\rangle|^2$$

is injective. Here, for $x \in \mathbb{F}^M$, $[x]$ denotes the equivalence class $\{\alpha x : \alpha \in \mathbb{F}, |\alpha| = 1\}$. Again the quadratic equations were the bridge to algebraic geometry. The injectivity is equivalent to the non-existence of rank-two matrices Q such that the Hilbert-Schmidt inner product between Q and all rank-one Hermitians $\{f_j \otimes f_j^*\}$ vanishes. The study of the variety of matrices with rank at most two was central to progress made on this problem at the workshop.

Activities. The typical daily schedule started with overview presentations in the mornings that had the goal of bridging the language barrier between the two groups of researchers, frame theorists and algebraic geometers. In the afternoon the open problems were rated and discussed in smaller groups. At the end of the workshop there was a concluding session in which promising challenging problems were identified. In the following we review the main issues that were discussed in these groups.

Equiangular tight frames. One problem studied in a group during the full week were questions about existence (and uniqueness) of real or complex equiangular tight frames and equiangular line systems respectively. Besides known results and constructions the main new contribution were various formulations of these problems as deciding whether suitable solution sets of affine algebraic varieties are not empty. Using Hilbert's Nullstellensatz the questions related to the existence of various objects could be reformulated as ideal membership problems.

More precisely, most of the problems could be reduced to the question whether the polynomial 1 is an element of a suitable ideal in a polynomial ring. While over the field of complex numbers this is a consequence of the Nullstellensatz, such reasonings give over the reals usually only necessary conditions whether varieties are not empty. One main contribution during the week was a reformulation of the real case (in many situations) using systems of polynomials which guarantee that among all complex solutions there has to be a real one. Hence also in these cases the Nullstellensatz could be applied to formulate equivalent ideal membership problems.

With this new approach to consider the existence of equiangular tight frames or related objects, it was for example possible to use Gröbner basis techniques to study these kind of problems. In particular, highly specialized software systems like Macaulay 2 or CoCoA were used to verify existence results in low dimensions. Optimizing the source code used in the computations and using more powerful machines could lead to new explicit cases were the existence questions can be answered. Any kind of useful information of the considered Gröbner bases or related algebraic objects would lead to further progress.

Connectedness of equal-norm tight frames. Another group studied the question whether the set of equal-norm tight frames is connected. Compared to equiangular tight frames, they satisfy a much weaker condition because apart from tightness, only the existence of $b > 0$ with $\|f_j\| = b$ for all j is required. Essential progress was made by using matrix complementation techniques together with a parametrization of the Klyachko-Horn inequalities and the honeycomb structure which implicitly classifies all equal-norm tight frames.

Phase retrieval. Phase retrieval was the topic in several groups, and in particular the question of the minimally possible number of elements in a frame, so that the operator, which maps a vector to the absolute value of its frame coefficients is injective. Observe that injectivity is a necessary conditions for phase retrieval to be possible.

One of the claims studied was that a generic choice of $4M - 4$ frame vectors gives an injective map A. In this case, generic means up to sets that are solutions of polynomial equations. Apart from the above-described formulation in terms of low-rank matrices, this was studied by pairing the frame vectors with possible rank-two matrices, considering the frame vectors over the same rank-two matrix as a fiber and then applying dimension counting arguments.

Similarly, the claim that $4M-5$ frame vectors can never provide injective maps was investigated. Essential for progress on this topic was the complexification technique and the degree of an algebraic variety. An odd degree established the existence of real solutions, which was formulated in terms of combinatorial expressions. Preliminary inspection showed that, apart from some number theoretic identity, the claim is true for $M = 2^k + 1, k \in \mathbb{N}$.

Another group studied the same question whether $4M-5$ frame vectors are not sufficient for injectivity with a different approach. This claim was transferred into a claim about the existence – or non-existence – of particular arrangements of subspaces in the real or complex Euclidean space. It was observed that this equivalent claim now allows methodologies from algebraic geometry to be applied. One possible direction is to utilize Schubert calculus, which in fact studies and provides results for such configurations to exist. The extensive theoretical framework of Schubert calculus provides a particularly promising novel attack to this question at the core of phase retrieval.

Summary. During the workshop two outstanding problems were resolved, the sufficiency of 4M-4 generic measurements for injectivity and connectedness of equal-norm tight frames, and significant progress was made on the construction and classification of equiangular tight frames in low dimensions as well as on the necessary size of a frame to provide phase retrieval. In all of these topics, success came from the rapidly evolving discussion between researchers from frame theory and from algebraic geometry.