Rigidity properties of free-by-cyclic groups

organized by

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Workshop Summary

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Organizers' Summary

A theorem of Stallings says that if M is a compact three-dimensional manifold, there exists a fibration $M \to S^1$ (whose fibers are surfaces) if and only if there exists a surjection $\pi_1(M) \to \mathbb{Z}$ whose kernel is finitely generated (and hence isomorphic to the fundamental group of a surface). When M has boundary, the surface does too, and its fundamental group is thus a free group. Because $\mathbb{Z}$ is free, we obtain algebraic information about $\pi_1(M)$: it is a semidirect product of the kernel (a free group in the case of boundary) with $\mathbb{Z}$; we describe this compactly as saying that $\pi_1(M)$ is "free-by-cyclic". Free-by-cyclic groups thus are a kind of generalization of fundamental groups of fibered 3-manifolds—at least when the manifold has boundary. While we understand fibered 3-manifolds fairly well, general free-by-cyclic groups remain mysterious.

The workshop was organized around the structure of free-by-cyclic groups, mostly guided by what we know about fibered 3-manifolds. Participants ranged from experts in various aspects of free-by-cyclic groups or 3-manifolds to those just getting started with the area. The morning talks were arranged to introduce:

(1) free-by-cylic groups;
(2) their fibered face theory;
(3) quasi-isometric rigidity;
(4) L^2-torsion;
(5) relative train track maps;
(6) subgroup separability;
(7) profinite rigidity;
(8) BNS-invariants;
(9) Cannon–Thurston maps; and
(10) outer automorphism groups of free-by-cyclic groups.

On occasion, the analogous aspects for 3-manifolds, along with the related difficulty of approaching free-by-cyclic groups, were discussed.

Monday's problem session was moderated by Kasia Jankiewicz and it concluded with 50 contributed problems! Over the course of the week, these problems were distilled down to 7 different working groups. We have included the summaries from the individual groups below. A few of the groups met all days of the week and there was some migration of

[1]MSC Classes: 20F65, 20F67, 20E08, 37C85, 57M07,

participants within the groups; AIM's workshop format facilitated this and we would like to thank the administrative staff for guiding us through the voting procedure.

Groups' summaries

Poset of laminations.

Our group worked on showing that the poset of attracting laminations associated to a monodromy of a free-by-cyclic group was actually a group invariant. An important step seemed to be characterizing periodic (with respect to the monodromy) malnormal subgroups in the fiber. With so many experts on currents in the group, we proved such a characterization using currents! It seems we are really close to showing group invariance of the depth of the poset.

QI-invariance of irreducibility.

In this discussion group, we considered the following question: let G be a free-by-cyclic group where the monodromy ϕ is reducible and let G' be a free-by-cyclic group where the monodromy ψ is irreducible. Can G be quasi-isometric to G'? Let $G = F_n \rtimes_\phi \mathbb{Z}$, $A \leq F_n$ be a subgroup invariant under ϕ, and H the free-by-cylic subgroup $A \rtimes_\phi \mathbb{Z}$. Now suppose there is a quasi-isometry f from G to $G' = F_m \rtimes_\psi \mathbb{Z}$. We tried to understand how $f(H)$ looks like in G' by examining the intersection of $f(H)$ with fibers in G'.

Consider the Menger curve $C = \partial f(\partial H)$ in the boundary of G'. Let $Ch(C)$ be the convex hull of C in G'. Our conjecture was that there is a dichotomy: either the intersection with the fiber F_m is a highly disconnected subset of $Ch(C)$, or $Ch(C)$ is coarsely invariant under the flow coming from ψ. The conclusion that $f(H)$ is flow-invariant would then imply that the monodromy ψ of G' is not irreducible.

Cubulating certain free-by-cyclic groups.

Our group investigated a pair of polynomially-growing automorphisms. They came from an infinite family whose mapping tori Rylee Lyman proved to be CAT(0) groups. The automorphisms are very simple: if $F(a, b, c)$ denotes a free group of rank three, the first sends a to a, b to aba and c to $aacaa$. The second sends a to a, b to aba and c to bcb. We sketched a proof, by arguing that certain axes would have to be "cubically straight", that the first automorphism cannot act geometrically on a CAT(0) cube complex. For the second automorphism, we studied a candidate cubulation. It has three-dimensional cubes, but we're currently hopeful that it will be cocompact.

LERFness of free-by-cyclic groups.

When are free-by-cyclic groups LERF? This is a question that has been of interest to geometric group theorists for some time, in different guises. During the workshop we considered the specific case of free-by-cyclic groups that are hyperbolic relative to a non-empty collection of $\mathbb{Z}^2$ subgroups. We discussed an attack on the problem using relative train-track representatives of the monodromy and were able to resolve the â€œgeometricâ€ cases of the question. Finally we connected the non-geometric but reducible cases to work by Hongbin Sun on non-LERF cyclic amalgams of 3-manifold groups. It seems possible to adapt Sunâ€™s method to the free-by-cyclic case with judicious use of fibered-face theory.

L^2-torsion of free-by-cyclic groups.

We discussed a strategy involving how to prove the "Chain Flare Condition" in certain cases. Namely, for "lone axis" outer automorphisms where there is a single illegal turn that never degenerates, which also do not contain any Nielsen paths. For a warm-up problem, we considered not the L^2-norm of chains, but the volume of the convex hull of a finite set of points. It seems like a "Brinkmann"-style argument should work. We sketched something out, but there are many details that need to be worked out.

Distorted subgroups in free-by-cyclic groups.
The group studying distorted subgroups in free-by-cyclic groups had a very productive series of sessions (M-F) trying to understand the problem: "Characterize the distorted subgroups of free-by-cyclic groups." For clarity, we restricted ourselves to the hyperbolic free-by-cyclic groups defined by irreducible automorphisms of F_n; the situation is very interesting. We studied an example by Barnard and Brady of a surface subgroup H in a free-by-cyclic group G which is distorted. This distortion comes from an (infinite index) subgroup of the free group in a presentation of $G = F_n \rtimes \mathbb{Z}$ given by an automorphism ϕ such that part of F_n is invariant under ϕ.

We settled on the following conjectural characterization: Let G be a hyperbolic free-by-cyclic group with irreducible monodromy. Then if H is a distorted subgroup of G, there is a subgroup H' of H that is a fiber or a semi-fiber of a subgroup G' of G. Both G' and H' could be of infinite index in H and G. The progress we made on the proof was using Feighn–Handel to get a good presentation of H, understanding how far the proof of the analogous statement for 3-manifolds could be pushed (this is mostly what we discussed), and looking at *stacks* (work of Bowditch) to understand the geometry of the universal cover.

Surface subgroups in free-by-cyclic groups.
We spent most of the time trying to understand how one can potentially adapt the Feighn-Handel proof of coherence of free-by-cyclic groups for producing new types of examples of non-free finitely generated subgroups of infinite index in such groups. We partially understood what sort of conditions potential examples of â€œminimal relative rank 1â€ in the context of their proof would have to satisfy to produce non-free subgroups, but it seemed hard to realize these conditions in practice, particularly for the fully irreducible monodromy of G. We also looked at the Barnard–Brady construction of surface subgroups in free-by-cyclic groups based on using NPC complexes with circle-valued Morse functions. This construction seems more promising and versatile in terms of constructing various other types of examples of subgroups, including non-surface ones, but we didnâ€™t have time to digest it properly.

Outcome

This was an extremely productive and stimulating week for the participants. The working groups delved into some exciting and challenging problems, generating numerous new ideas. All the participants learned a wide array of techniques and tools, and at least one group achieved substantial progress on their problem over the week. New collaborations were established, and we anticipate these discussions to persist in the future.

Bibliography