Functoriality and the trace formula

organized by
Ali Altug, James Arthur, Bill Casselman, and Tasho Kaletha

Workshop Summary

This workshop was devoted to exploring the future of Langlands' functoriality conjectures relating the parametrization of automorphic forms to automorphic L-functions. It was the third in a series of workshops on this topic in the past few years, and the first held at AIM.

For the organizers, the main idea was to cover what role the Trace Formula might play, but other approaches were also brought in, most notably the one introduced in a series of papers by Braverman and Kazhdan. A third stream is that related to spherical varieties and the relative trace formula.

A preliminary outline, with reading suggestions and a few additions made after the workshop, can be found at

`http://www.math.ubc.ca/~cass/aim-reading.html`

This contains also a list of participants.

Morning sessions

The morning sessions consisted of two one and a half hour talks on each of Monday through Friday. They were:

- Monday: Ali Altug—A Brief Overview of Beyond Endoscopy
  James Arthur—L-functions, Functoriality and the Trace Formula
- Tuesday: Bao Chau Ngo—Hankel transform and Langlands functoriality
  Colette Moeglin—A-packets and highest weight modules
- Wednesday: Yiannis Sakellaridis—Transfer operators between relative trace formulas in rank one
  Freydoon Shahidi—On Fourier transforms of Braverman and Kazhdan
- Thursday: Wei Zhang—Gross-Zagier for function fields
  Zhiwei Yun—Orbital integrals and Dedekind zeta functions
- Friday: Diana Shelstad—Beyond Endoscopy: an approach to stable-stable transfer at the archimedean places
  James Arthur—The long view a.k.a. The Big Picture

Afternoon sessions

The number of participants was relatively large for an AIM workshop. Because the group was so large, there were several afternoon sessions, which could be described roughly as expanding on the morning talks. Some of these interpolated material, for the benefit of non-experts, while others looked at problems raised elsewhere. These afternoon sessions included:

- ∘ Trace formula for $GL(2)$ and the analytic issues regarding the Fourier transform, relative trace formula and non-standard Schwartz spaces (a la Sakellaridis) (Altug-Sakellaridis)
- ∘ the approach of Braverman-Kazhdan (Ngo-Shahidi)
- ∘ Geometry of orbital integrals and counting points on Springer fibers (Tsai-Yun)
- ∘ Arthur packets (Moeglin)
- ∘ Motivic integration and orbital integrals (Gordon)
- ∘ Beyond endoscopy and families of automorphic representations (Altug-Getz)
- ∘ Trace formula for $PGL(2)$ via a regularization (Sakellaridis)

Via Trace Formulas

There are roughly two major approaches being tried at the moment. One follows closely ideas introduced by Langlands in the early 2000s and developed by him, partly in collaboration with Frenkel and Ngo, over the past 20 years. It exploits the Stable Trace Formula in an essential way. As such, it assumes all of Arthur's work on the invariance and stabilization of the trace formula, and proposes to go much further in the fine analysis of this formula. This makes the material very difficult, both because the required background is vast, and because the technical challenges are daunting.

The morning talks of Arthur and Altug addressed this idea. Altug gave a careful exposition of the development of Langlands' ideas. These begin with a paper from early 2000s, which introduced the use of the stable trace formula in a non-standard way in order to detect the functorial source of automorphic representations by investigating the residues of poles of the corresponding L-function as weights on the spectral side. This paper prompted a letter from Sarnak pointing out a number of challenges coming from analytic number theory, and possible alternative strategies. The matter was further developed by Langlands together with Frenkel and Ngo in a paper from 2010, which introduces the Steinberg-Hitchin base, which may be identified with a real affine space, and the use of the Fourier transform on it as a method to isolate the contribution of the most dominant, namely the trivial, representation. Arthur continued this line of thought and gave a description of Altug's work that substantially deepens these ideas and brings them to fruition, in the special case of the group GL_2, where a number of the severe analytic difficulties can be handled, still with much effort and ingenuity, by brute force.

The other morning talks devoted to this approach were research-oriented. A generalization of the Ramanujan conjecture due to Arthur proposes that only certain automorphic forms called now A-packets can occur in the spectra of arithmetic quotients. In order for the stable trace formula to be used for Langlands' proposals about functoriality, these non-tempered representations are to be stripped away, in an inductive fashion, in order to arrive at the tempered ones. For this, an understanding of the properties of A-packets is essential. They have proved to be surprisingly complicated. In a morning talk, Moeglin discussed recent results on archimedean Arthur-packets. She has studied Arthur packets for both real and p-adic groups and has proved a number of important properties. In her talk, she focused on the case of classical real groups and described her results towards the problem of determining, for a given irreducible representation π, the set of all Arthur parameters whose corresponding packets contain π.

The topic of Arthur packets was further taken up in an afternoon session chaired by Moeglin. She explained various properties that hold for L-packets but fail for A-packets, and gave counterexamples in each case. This was a very valuable lecture, as Moeglin is possibly the only person in the world with such detailed knowledge on the still very elusive nature of A-packets.

The trace formula and its application to functoriality was further discussed in a morning talk by Sakellaridis in the setting of spherical varieties—i.e. the relative trace formula—that substantially generalizes that of reductive groups. Sakellaridis discussed functorial transfer between spherical varieties and its application to automorphic periods and L-functions. While this more general setting is even harder to fully understand than the already highly complicated situation with reductive groups, it offers a valuable new point of view, as well as simple test cases that are unavailable when one works with just reductive groups. One such case occurs when one considers spherical varieties with the same Langlands dual group. In the reductive group case, this implies that the groups are inner forms of each other, and hence the situation is rather simple. But in the case of spherical varieties, this offers a set-up which avoids some serious technical complications, and yet is fruitful enough to allow the exploration of some important phenomena. Sakellaridis specialized even further to the case of varieties of rank 1 and explained his results on the existence of transfer operators, generalizing the endoscopic transfer relations for reductive groups.

This topic was further discussed in an afternoon session chaired by Sakellaridis. It consisted of group discussions about various techniques and approaches one could apply to push the results beyond the case of rank 1. Some preliminary results were presented by Chen Wan for the groups GL_2 and GL_3, which offered explicit formulas for transfer operators.

The question of transfer operators was taken up, in a different form, in Shelstad's morning talk. The setting now is that one considers only reductive groups, but drops the equality of L-groups. Again, the question is to find transfer operators between appropriate spaces of Schwartz functions. These are the local input to the comparison of trace formulas between these groups. Shelstad explained the general setting. She drew attention to certain formulas of Harish-Chandra about the Fourier transforms of the characters of discrete series representations of the group SL_2, and then proceeded to explain how these formulas suggest an approach to the proof of the so called stable-stable transfer for real groups.

The matter of the relative trace formula was further discussed in the morning talk of Wei Zhang, who explained how in the setting of function fields the trace formula computes the intersection of certain cycles on spaces of Shtukas. The result of this comparison is a generalization of the celebrated theorem of Gross-Zagier to arbitrary high derivatives of L-functions.

The geometric interpretation of orbital integrals, which is one of the underpinnings of Zhang's talk, is a central theme in geometric representation theory. It was also present in Zhiwei Yun's talk, which discussed the Dedekind ζ-functions of orders, as well as in the afternoon session chaired by Yun, where these ideas were further explored, and recent results of Chen-Chiang Tsai on the asymptotic behavior of orbital integrals were discussed.

The organizers made sure to have one afternoon session, chaired by Arthur, that introduced Langlands' ideas on Beyond Endoscopy in a more basic way, aimed at graduate students interested in the field.

Via L-functions

The other main approach is that starting with work of Braverman and Kazhdan, which hopes to first derive properties of L-functions directly, and then perhaps find consequences for functoriality and then to establish (many cases of) functoriality through various converse theorems. This is, roughly speaking, what Ngo and Shahidi talked about, and there were many fruitful afternoon sessions on this topic.

Ngo started by giving a general description of the ideas of Braverman and Kazhdan, generalizing the work of Godement-Jacquet. In the work of Godement-Jacquet one writes an integral representation for the standard L-function of an automorphic representation on $GL(n)$ by integrating an appropriate function on the space of $n \times n$ matrices. One then shows the analytic continuation and functional equation of the L-function using Poisson, following Tate's thesis. The essence of the Braverman-Kazhdan idea is to generalize this to general automorphic L-functions of any group G. This generalization comes with a series of problems and conjectures: one needs to find a generalization of the space of matrices, the notion of an "appropriate function", and Poisson summation. The proper generalization of the space of matrices is found in Vinberg's theory of monoids, which is mor or less attached to the data defining the automorphic L-function (i.e. a reductive group G and a dual group representation). Braverman and Kazhdan then conjectures the existence of a "Schwartz space" and a "Fourier-like" transform that satisfies a "Poisson-like summation" formula, which would give the analytic continuation of the automorphic L-functions. The Fourier-like transform is expected to be defined locally but on the other hand the Poisson-like summation formula would be global.

Ngo's morning talk was on his very recent construction of the local Fourier-like transform described above (Ngo calls this the "Hankel transform" bearing the analogy with the classical Hankel transform). He started with explained the theory of the monoids mentioned above and then described his construction of the Hankel transform starting with tori.

The problem of Poisson-like summation was taken up in Shahidi's morning talk, where he described how the doubling method can be interpreted in the context of Braverman-Kazhdan (which also was considered by Braverman and Kazhdan themselves). He then described how the theory of intertwining operators can be normalized suitably to give a Fourier-like transform, and the theory of Eisenstein series in this context provide the analogue of Poisson summation formula.

The ideas about the Braverman-Kazhdan proposals led to fruitful discussions in the afternoon sessions led by Ngo. The predominant problem that was discussed throughout the afternoon sessions was whether the Fourier-like transform defined via intertwining operators, described by Shahidi—a lift of certain normalized intertwining operators originally introduced by Piatetski-Shapiro and Rallis—in the context of the doubling method. During the workshop a plausible argument was sketched, showing that the defining kernel of this transform agrees with that of the corresponding Hankel transform of Ngo which is now defined very generally. Details are now being verified—i.e. since the workshop—including the correctness of the definition of Schwartz spaces in many cases.

Summary of accomplishments

The workshop seems to have been judged by most participants to be successful, although the principal progress was made in a way unanticipated by the organizers. This was in understanding better the significance of the proposals of Braverman and Kazhdan. In the period since the workshop, there has been a great deal of further activity by Getz, Ngo, Jiang, and Shahidi along these lines. A summary of the present state of affairs has been written up by Ngo in the preprint *Hankel transform, Langlands functoriality, and functional equation of automorphic L-functions* available on the web page

`http://www.math.ubc.ca/~cass/aim-reading.html`

The organizers would have liked to have seen more progress on understanding the role of the Trace Formula in verifying functoriality. They expected also some insight into how the approaches of Langlands and Braverman-Kazhdan are related. While some new ideas were elucidated, especially concerning the relative trace formula, the general structure still remains fairly vague.

The organizers were pleased to see that several of the younger people have been definitely motivated to pursue things further.