Classifying fusion categories

organized by
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Workshop Summary

The main purpose of the AIM workshop *Classifying Fusion Categories* was to bring together researchers from the various fields in which the classification problem is studied in order to share results and techniques with a view towards a long-term classification program.

The participants mostly came from the areas of Representation Theory (of Hopf algebras and their generalizations), Fusion Categories and Subfactor Theory. Many participants had only previously met a small subset of the other participants. The participants included 4 women, 4 researchers from South America, 2 from Europe, 1 from Japan and 1 from Israel. A total of 5 graduate students participated.

The usual AIM format was followed: most mornings there were 2 talks and the afternoons included both large group discussions and small break-out sessions. One afternoon was dedicated to compiling a list of open problems.

Talks were given by Dmitri Nikshych and Noah Snyder (introduction to fusion categories and translating between subfactor and fusion categories), Michael Mueger and Richard Ng (on non-degenerate/modular categories), Scott Morrison, Dave Penneys and Emily Peters (planar algebra approaches and examples), David Jordan and Sonia Natale (extension theory) and Zhenghan Wang (applications of modular categories to quantum systems and computation).

The main theme of the workshop was classification of fusion categories that are "small" in various senses, subject to various qualitative constraints. The acceptable precision of the classification was also open to interpretation. A somewhat vague secondary theme was the question: *do we know where all fusion categories come from?* Underlying this theme were several specific computational questions on how to determine various structures and statistics from the basic data of a fusion category.

The depth of our current understanding of "small" fusion categories is (roughly) jointly proportional to the amount of structure and algebraic properties of the FP-dimension of simple objects. In terms of structure, it is easier to classify (small) modular categories than finite ribbon categories or fusion categories that are not assumed to be braided at all. In terms of dimensions, we can completely classify fusion categories in which all simple objects are 1-dimensional and it is known that any fusion category whose objects are integer-dimensional is equivalent to $Rep(A)$ for a quasi-Hopf algebra A, but the problem of understanding fusion categories whose dimensions lie in $\mathbb{Z}[\sqrt{2}]$ is completely open.

The first day the participants studied various versions of classifying fusion categories with small global (FP) dimension. For example, one group specialized to the case of integral modular categories, while another considered general fusion categories with small FP-dimension and small rank. Later, participants of this group classified possible fusion rings

for fusion categories with global dimension less than 30, under the additional hypothesis that the category is generated by a self-dual object with strictly smallest dimension (other than 1). Another group discussed Izumi's approach to classifying near-group categories. Izumi is intending to publish his results on this approach soon. Another group made substantial progress towards classifying certain fusion categories with objects of dimension less than 2 which occur as extensions of even parts of ADE subfactors.

The second day was partially dedicated to two disparate views on the classification of modular categories. Recent number-theoretic results of of Ng (some with Schauenburg) have led to new approaches to the classification of low rank modular categories. Müger, Davydov, Nikshych and collaborators have developed the idea of the Witt group for non-degenerate fusion categories (i.e. modular except without assuming a spherical structure). This is a somewhat coarse approach: the trivial Witt class includes all Drinfeld centers of spherical categories, the main source of "exotic" modular categories. This leads to a precise formulation of a conjecture of Moore and Seiberg: *the modular categories obtained from quantum groups at roots of unity generate the Witt group*. The Frobenius-Schur indicator for pivotal categories plays an important role in Ng's work. In particular, it was previously known that dimensions of objects in fusion categories are integers in cyclotomic fields, however it was not clear where the particular roots of unity came from. In Ng's approach he shows that one can use only nth roots of unity where n is the Frobenius-Schur exponent. This is closely related to the use of rotations in the study of subfactor planar algebras (pioneered by Jones), and thus was of great interest among the subfactor community.

The morning of the third day concentrated on techniques coming from subfactors and planar algebras; talks by Emily Peters, Scott Morrison and David Penneys covered the known constructions of exotic subfactors, and techniques coming from analyzing quadratic tangles in planar algebras which give strong restrictions on possible fusion rings. In the afternoon the participants compiled a list of open problems through a moderated problem session led by Gelaki. The problem list is available on the AIM website http://aimpl.org/fusioncat and we hope to post it at arxiv.org later. The list includes rather specific problems as well as more general questions. During that afternoon, we also prepared a 'zoo' of all known small examples and constructions of fusion categories, modular tensor categories, and subfactors.

The 4th and 5th days followed the same basic format as previous days: talks in the mornings and break-out problem sessions in the afternoons. Progress reports on the problem sessions were presented to the larger group, typically in the afternoon before the smaller groups were formed. The problem sessions on these days did not focus on the subject of the mornings talks, but often continued on subjects treated earlier in the week.

Talks on the fourth day covered extension theory in various aspects. Sonia Natale spoke about short exact sequences of Hopf algebras and fusion categories, and David Jordan explained the theory of extension of a fusion category by a finite group due to Etingof, Nikshych and Ostrik. He illustrated this by working out the classification of Tambara-Yamagami categories. On Friday Zhenghan Wang gave an overview of the relationship between topological quantum field theory and quantum computing, discussing in detail the Ising theory and Kitaev's toric code.

We will give a couple of representative examples of newproblems that were worked on during the last two days. Given a fusion category $\mathcal{C}$ there is a modular category attached to it called the Drinfeld center $Z(\mathcal{C})$. Many arguments go through the Drinfel'd center, but computing it in practice is somewhat difficult. One group worked on computing the Drinfel'd

center of the exceptional fusion categories coming from the Asaeda-Haagerup subfactor. Although the complete calculation of this center appeared difficult, we were able to make some progress, for example restricting which dimensions of objects are possible and explicitly identifying the dimensions of several objects. One fascinating question which Jones asked during the problem session was whether for non-pointed fusion categories there's an absolute bound on how large of n you can have such that $V^{\otimes n}$ is simple. This is a modification of the "supertransitivity" notion for subfactors and is inspired by the classification of highly transitive group actions. Snyder gave an example where $n = 2$, and Bisch, Haagerup, and Izumi found an example with $n = 3$. Bisch and Haagerup have an approach which may give larger n as well.

Several groups elected to continue working on their problems via email after the workshop. For example one large group (6 or more participants) has started to write a paper on classification of integral modular categories $\mathcal{C}$ with $FPdim(\mathcal{C})$ having few prime divisors. One goal is to determine conditions under which non-group-theoretical examples can exist. Another group is working on the classification of low rank modular categories, with a view towards understanding which Galois groups can occur, and proving Wang's conjecture (finitely many modular categories of fixed rank). This group has expressed interest in applying to the SQuaRES program. A group is continuing work on constructing one of the missing subfactors in Grossman and Snyder's analysis of the Morita equivalence class of the Asaeda-Haagerup subfactor. They are attacking this using several different approaches, the most promising based on Izumi's work on near-group categories.

Some of the participants expect to propose a session at the 1st Mathematical Congress of the Americas (2013) as a follow-up, and other follow-up workshops/conferences are anticipated. Many of the participants are attending "Subfactors in Maui" during July 2012, the organizers there have extended several extra invitations based on collaborations started at AIM.