New directions in G2 geometry

organized by
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Workshop Summary

Introduction.
The primary goals of this workshop, concerning the field of G_2 geometry in 7 dimensions and related areas, were (i) advertise and reconsider select results in the literature which seem to have not yet been developed to their full potential, (ii) suggest and explore brand new research directions.

The topic of G_2 geometry has grown substantially in recent years and now comprises a large network of researchers with varying backgrounds across geometry, topology and mathematical physics. However, much of the focus of the field has been on certain specific aspects of G_2 geometry, such as the construction of new holonomy G_2 manifolds, calibrated submanifolds and special connections (G_2 instantons and monopoles), and the pursuit of possible geometric invariants inspired by the programme of Donaldson–Thomas.

This workshop focussed instead on lesser-studied aspects of G_2 geometry, chosen both because of their intrinsic interest and because of their expected potential to lead to substantially new avenues for research. Specifically, the main themes for the workshop were as follows.

- New directions in G_2 mirror symmetry.
- Topological methods and formality in G_2 geometry.
- Loop spaces and non-associative gauge theory.
- G_2 function theory and G_2 mappings.

Summary.
During the workshop each of the main themes was explored, along with other topics, by a diverse group of researchers with a range of different expertise. Notably, there was a significant number of early career researchers, including many graduate students and postdocs, which contributed to a very lively and dynamic environment throughout the workshop.

Participants, including speakers, were encouraged by the organizers to be bold and radical throughout the workshop: this led to a free exchange of ideas and fascinating speculation, and drove significant progress towards various research questions. As well as drawing up a list of problems at the start of the workshop, we had a final session where we created another list of problems for the future: both of these lists will be extremely valuable resources for the community.

Problem sessions.

The problem sessions proved to be extremely stimulating and productive, with a wide variety of topics pursued.

Rational homotopy theory and G_2 structures. The group discussed various ways in which methods from rational homotopy theory could be used to understand aspects of G_2 geeometry. Though ideas are currently quite vague, the group intends to continue to pursue this and look more into the details.

G_2 bordisms. The group looked at a cohomogeneity-one approach to finding a closed G_2 bordism between the nearly Kähler 6-sphere and itself with the opposite orientation. The group intends to continue to examine this situation to determine whether or not such a closed G_2 bordism can exist.

Geometric structures on loop spaces. The group found some interesting structures on loop spaces arising whenever there is a calibration on the underlying manifold. In particular, for G_2 manifolds there was the possibility of studying the Fukaya category of the loop space as a potentially natural and useful object.

Linking number for associatives. The group arrived at a definition of local linking number for associatives and showed explicitly that it can be nontrivial. The group plans to write up this work and hopes to describe the relationship between this invariant and the boundary structure of the moduli spaces of associatives.

G_2 mirrors. This group combined the construction of associatives by Dwivedi–Platt–Walpuski with the version of mirror symmetry achieved using the real Fourier–Mukai transform, to give a promising approach to constructing the first non-trivial examples of deformed G_2-instantons on compact holonomy G_2 manifolds. The group intends to pursue this exciting line of research.

Higher degree pluripotential theory. Harvey–Lawson define a new differential operator on functions in the presence of a calibration, which is the basis for their pluripotential theory. The group studied the possibility of extending this operator to higher-degree forms and has found promising candidates, which it plans to study further.

Pluripotential theory and cohomology. The group studied the possibility of using Harvey–Lawson pluripotential theory to construct new cohomology classes á la Chern–Weil theory, but this ultimately proved to be unsuccessful. However, there are still interesting directions here to be explored.

Geometry of G_2 moduli space. The group considered several directions regarding the moduli space of torsion-free G_2-structures. They looked at special points related to a distinguished symmetric cubic on the moduli space or to smallest volume associatives/coassociatives. They also studied the immersion as a Lagrangian in the product $H^3 \times H^4$ of cohomology groups. All of these directions are worthy of further consideration.

Kovalev–Lefschetz fibrations. The group looked at two distinct ideas related to G_2 manifolds arising as Kovalev–Lefschetz fibrations. The first was to see how the moduli space of $K3$ surfaces plays a role in the G_2 moduli space. The second was to understand how the topology of the G_2 manifold relates to the link of singular fibres. Both directions will be very interesting for the community and will be actively pursued.

Monopoles. The group outlined a promising new construction for Calabi–Yau monopoles when the Calabi–Yau was fibred by K3 surfaces. The group plans to continue working to realize this construction and to see if this construction can be adapted or be used to build G_2 monopoles with singularities.

G_2-Laplacian flow and degenerate initial data. The group considered starting the G_2-Laplacian flow at closed 3-forms which define a quadratic form that is only non-negative. The group explored a cohomogeneity-one setting but this proved to be unsuccessful, as any non-zero but non-negative invariant 3-form had to be positive. However, the setting of a T^4-fibration, which led to space-like mean curvature flow (with some lightlike degeneracy in the initial condition) seemed much more promising and worthy of further study.

Monotone quantities for G_2 flows. The group considered a wide variety of possible monotone quantities in a range of settings. One direction studied more thoroughly was the case of dimensional reductions, particularly the setting of a coassociative fibration for the G_2-Laplacian flow. Although no useful quantities were identified, there are many avenues still to be explored here.

G_2 analogues of the Calabi conjecture. The group pursued various directions here, including prescribing the volume form induced by the G_2 structure, and a variation on the open question of uniqueness of torsion-free G_2-structures in a given cohomology class, related to pluripotential theory. Whilst no concrete plans to continue were made, the research directions considered here are certainly worth pursuing further.

Conclusion.
The organizers were very pleased with the outcome of the workshop. It clearly achieved both its goals, in particular instigating entirely novel research directions in G_2 geometry, with clear progress on various avenues that will be pursued by several new collaborative teams. It is probable that we shall see tangible, significant, results as a direct consequence of this workshop in the near future.

The organizers are sincerely grateful to the AIM staff for providing support for this very successful workshop.