

OPEN PROBLEMS AND QUESTIONS ABOUT GEODESICS

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ABSTRACT. The paper surveys open problems and questions related to geodesics defined by Riemannian, Finsler, semi Riemannian and magnetic structures on manifolds. It is an extended report on problem sessions held during the International Workshop on Geodesics in August 2010 at the Chern Institute of Mathematics in Tianjin.

This paper is an extended report on problem sessions held during the International Workshop on Geodesics in August 2010 at the Chern Institute of Mathematics in Tianjin. The focus of the conference was on geodesics in smooth manifolds. It was organized by Victor Bangert and Yiming Long and supported by AIM, CIM, and NSF.

1. NOTATION AND DEFINITIONS

In this paper M is a connected C^∞ manifold and \widetilde{M} is its universal cover. We consider various structures on M that create geodesics: Riemannian (later also semi-Riemannian) metric g , Finsler metric F , magnetic structure ω . The unit tangent bundle defined by such a structure is denoted by SM and ϕ_t is the geodesic flow on SM . The geodesic defined by a vector $v \in SM$ is denoted by γ_v ; it is parametrized by arclength unless otherwise specified.

The free loop space ΛM is the set of all piecewise smooth mappings from the circle S^1 to the manifold equipped with the natural topology. When M has a Riemannian metric (or other suitable structure) we can consider the subspace $\Lambda_T M$ of loops with length $\leq T$. Similarly given two points $p, q \in M$, we denote by $\Omega(p, q)$ the space of piecewise smooth paths from p to q and by $\Omega_T(p, q)$ the subspace of paths with length at most T .

We define the energy functional on ΛM by

$$c \mapsto \int_{S^1} \|\dot{c}(s)\|^2 d\lambda(s),$$

where $\|\cdot\|$ is the norm induced by a Riemannian metric or other suitable structure and λ is Lebesgue measure on S^1 normalized to be a probability measure. The critical points of this functional are the closed geodesics for the metric (parametrized at the constant speed $Length(c)$). In the case of $\Omega(p, q)$, the circle S^1 is replaced by the interval $[0, 1]$.

2. CLOSED GEODESICS

Given a Riemannian manifold, does there exist a closed geodesic? If yes, how many (geometrically different) closed geodesics must exist? Existence is known if the manifold is closed. For surfaces (and certain manifolds of higher dimension, e.g. S^n), the result is essentially due to Birkhoff [Birkhoff1927, Chap. V]. He used two arguments. The first is a variational argument in the free loop space ΛM . From

the subspace of ΛM consisting of all loops homotopic to a given loop, one selects a sequence of loops whose lengths converge to the infimum of the length function on the subspace. If the loops from this sequence lie in a compact region of the manifold (for example, if the manifold itself is compact), then by the Arzelà-Ascoli theorem there exists a loop where the infimum is achieved; if the infimum is not zero, this loop is automatically a nontrivial closed geodesic. These arguments show the existence of closed geodesics in many cases, in particular for compact manifolds with non trivial fundamental group.

If the infimum of the length of the loops homotopic to a given loop is zero, Birkhoff suggested another procedure (actually, a trick) to prove the existence of closed geodesics, which is explained for example in [Birkhoff1927, §§6,7 of Chap. V]. This trick is nowadays called the *Birkhoff minimax procedure*, and was used for example to show the existence of closed geodesics on any Riemannian sphere with dimension ≥ 2 .

Let us recall this procedure in the simplest case, when the manifold is S^2 . We consider the foliation of the sphere without two points (north and south poles) as in the picture: if we think of the standard embedding in \mathbb{R}^3 , the fibers are the intersections of the sphere with the planes $\{x_3 = \text{const}\}$.



The foliation of the sphere without south/north poles used in the Birkhoff minimax procedure

Now apply a curve shortening procedure to every curve of this foliation, for example the curvature flow (Birkhoff used another shortening procedure). We need that the evolution of the curves in this procedure depends continuously on the curve. We obtain a sequence F_i of the foliations of the sphere without two points into curves. For each foliation F_i of the sequence, let γ_i be a leaf of maximal length. Because of topology, the lengths of the circles γ_i are bounded from below by a certain positive number. By the Arzelà-Ascoli theorem, the sequence of curves γ_i has a convergent subsequence. The limit of this subsequence is a stable point of the shortening procedure, and is therefore a geodesic.

For arbitrary closed manifolds, the existence of a closed geodesic is due to Lyusternik and Fet ([Lyu-Fet1951] and [Fet1952]). They considered the energy functional on the loop space ΛM and showed that the topology of ΛM is complicated enough so that the energy functional must have critical points with nonzero energy (which are non trivial closed geodesics). Lyusternik and Schnirelmann [Lyu-Sch1934] had earlier used a related argument to show that any Riemannian metric on S^2 has at least 3 simple closed geodesics.

How many closed geodesics must exist on a closed manifold? For surfaces, the answer is known: there are always infinitely many geometrically different closed geodesics. This is easily proved using Birkhoff's first argument when the fundamental group is infinite. The remaining cases of the sphere and the projective plane were settled by [Bangert1993] and [Franks1992]. In higher dimensions Rademacher has

shown that a closed manifold with a generic Riemannian metric admits infinitely many geometrically different closed geodesics [Rademacher1989].

The main approach to proving the existence of infinitely many closed geodesics has been to apply Morse theory to the energy functional on the free loop space ΛM . The critical points of this functional are precisely the closed geodesics. But it has to be remembered that each closed geodesic can be traversed an arbitrary number of times. It is thus important to distinguish geometrically different geodesics from repetitions of the same geodesic. This distinction is difficult to make and so far, despite some published claims, the existence of infinitely many closed geodesics on a general compact Riemannian manifold has not been proven. Nor have the resources of the Morse theory approach been fully exhausted.

One can modify these classical questions in different directions; this will be done below.

2.1. Riemannian metrics on spheres. Except in dimension 2, all that is currently known for a general Riemannian metric on a sphere are the results that hold for all compact Riemannian manifolds. *Is there an extension of Lyusternik and Schnirelmann's result to higher dimensions, in particular to S^3 ?*

2.2. Finsler metrics on S^2 . The arguments of Fet (and Morse) can be adapted to the Finsler setting: one can show the existence of at least one closed geodesic on every compact manifold. Moreover, if the Finsler metric is reversible, or if the manifold is a surface other than the sphere or the projective plane, one can show the existence of infinitely many geometrically different closed geodesics.

The following example constructed in [Katok1973] shows that the number of closed geodesics on a 2-sphere with an irreversible metric can be 2. Consider the sphere S^2 with the standard metric g_0 of constant curvature, and a Killing vector field V on it. For small enough $\alpha \in \mathbb{R}$ one has the following Finsler metric, known as a Randers metric:

$$F(x, \xi) = \sqrt{g_0(\xi, \xi)} + \alpha g_0(V, \xi).$$

Katok has shown that for certain values of α there are precisely 2 closed geodesics; they are the unique great circle tangent to V , parametrized in both directions. This example can be generalized to higher dimensions: one obtains a Finsler metric on S^n with precisely $2\lfloor(n+1)/2\rfloor$ distinct prime closed geodesics; see [Katok1973].

V. Bangert and Y. Long in [Ban-Lon2010] and [Long2006] proved that there are always at least 2 distinct prime closed geodesics for every irreversible Finsler metric on S^2 .

Conjecture 2.2.1 (Long, Bangert, Problem 15 from [Álvarez2006]). *Every irreversible Finsler metric on S^2 has either exactly 2 or infinitely many distinct prime closed geodesics.*

There exist results supporting this conjecture. In particular, H. Hofer, K. Wysocki and E. Zehnder in [Ho-Wy-Ze2003] studied Reeb orbits on contact S^3 . Their result can be projected down to S^2 , and implies that the total number of distinct prime closed geodesics for a bumpy Finsler metric on S^2 is either 2 or infinite, provided the stable and unstable manifolds of every hyperbolic closed geodesics intersect transversally. See also A. Harris and G. Paternain in [Har-Pat2008].

The closed geodesics in Katok's example are elliptic.

Conjecture 2.2.2 (Long). *The existence of one hyperbolic prime closed geodesic on a Finsler S^2 implies the existence of infinitely many distinct prime closed geodesics.*

Conjecture 2.2.3 (Long). *Every Finsler S^2 has at least one elliptic prime closed geodesic.*

The conjecture agrees with a result of Y. Long and W. Wang who proved that there are always at least 2 elliptic prime closed geodesics on every irreversible Finsler S^2 , if the total number of prime closed geodesics is finite [Lon-Wan2008]. The conjecture does not contradict [Grjuntal1979] where an example of a metric such that all closed simple geodesic are hyperbolic is constructed. Indeed, for certain metrics on the sphere (and even for the metric of certain ellipsoids) most prime closed geodesics are not simple.

2.3. Of complete Riemannian metrics with finite volume.

Question 2.3.1 (Bangert). *Does every complete Riemannian manifold with finite volume have at least one closed geodesic?*

The question was answered affirmatively for dimension 2. Moreover, in dimension 2 a complete Riemannian manifold of finite volume even has infinitely many geometrically different geodesics [Bangert1980]. The argument that was used in the proof is based on the Birkhoff minimax procedure we recalled in the beginning of §2, and does not work in dimensions ≥ 3 . One can even hope to construct counterexamples in the class of Liouville-integrable geodesic flows. In this case, most orbits of the geodesic flow are rational or irrational windings on the Liouville tori; they are closed, if all of the corresponding frequencies are rational. Since there are essentially $(n - 1)$ frequencies in dimension n , one can hope that if $n > 2$ it would be possible to ensure that there is always at least one irrational frequency. Initial attempts to find a counterexample on $T^2 \times \mathbb{R}$ with the metric of the form $a(r)d\phi^2 + b(r)d\psi^2 + dr^2$ were, however, unsuccessful.

A more difficult problem would be to prove *that on any complete Riemannian manifold of finite volume there exist infinitely many closed geodesics.*

Questions of this nature are also interesting in the realm of Finsler geometry. One would expect the results for reversible Finsler metrics to be very similar to those for Riemannian metrics. For irreversible Finsler metrics, Katok’s example shows that there are compact Finsler manifolds with only finitely many closed geodesics, but it is still possible that all non compact Finsler manifolds with finite volume might have infinitely many closed geodesics. On the other hand, the following question is completely open:

Question 2.3.2 (Bangert). *Does there exist an irreversible Finsler metric of finite volume on $\mathbb{R} \times S^1$ with no closed geodesics?*

2.4. Of magnetic flows on closed surfaces. It is known that the trajectory of a charged particle in the presence of magnetic forces (=“magnetic geodesic”) is described by a Hamiltonian system with the “kinetic” Hamiltonian of the form $\sum_{i,j} p_i p_j g^{ij}$ on T^*M with the symplectic form $dp \wedge dx + \pi^* \omega$, where ω is a closed (but not necessarily exact) form on M and $\pi : T^*M \rightarrow M$ is the canonical projection.

We assume that our surface M^2 is closed.

Question 2.4.1 (Paternain). *Is there at least one closed magnetic geodesic in every energy level?*

If the form is *exact*, the affirmative answer was obtained by Contreras, Macarini and Paternain in [Co-Ma-Pa2004], which is partially based on [Taimanov1992].

It seems that the standard variational method to solve this problem does not work in this setting, because the form is not exact and therefore corresponds to no Lagrangian. One can try to solve this problem by applying a result of Hofer, Wysocki and Zehnder [Ho-Wy-Ze1993]. In view of this paper, it is sufficient to show that the flow is of contact type.

We also refer the reader to a survey [Ginzburg1996].

3. PATH AND LOOP SPACES

As noted in the previous section, one of the main approaches to proving the existence of closed geodesics is to use topological complexity of the loop space ΛM to force the existence of critical points of the energy functional. Loops with length $\leq T$ correspond to critical points in $\Lambda_T M$. Similarly geodesics joining two points p and q can be studied by investigating the path spaces $\Omega(p, q)$ or $\Omega_T(p, q)$. The homology of these spaces have been much studied.

3.1. Sums of the Betti numbers. Let p, q be points in a Riemannian manifold. The space $\Omega_T(p, q)$ of paths from p to q with length $\leq T$ has the homotopy type of a finite complex (see eg. [Milnor1963]), and hence the sum of its Betti numbers is finite for each T . The same is true for the space $\Lambda_T M$ of loops with length at most T .

Question 3.1.1 (Paternain). *How do the sums of the Betti numbers for $\Lambda_T M$ and $\Omega_T(p, q)$ grow as $T \rightarrow \infty$? Does the growth depend on the metric?*

It was shown by Gromov in [Gromov2001] that if M is simply connected there is a constant C such that sum of the Betti numbers of Λ_{CN} is at least $\sum_{i=0}^N b_i(\Lambda)$.

Question 3.1.2 (Gromov). *Does the number of closed geodesics with length $\leq T$ grow exponentially as $T \rightarrow \infty$ if the Betti numbers of the loop space grow exponentially?*

From [Gromov1978] it follows that the answer is positive for generic metrics.

Here is a potentially interesting example. Consider $f : S^3 \times S^3 \rightarrow S^3 \times S^3$ such that the induced action on $H_3(S^3 \times S^3)$ is hyperbolic. Let M be the mapping torus for f , i.e. $S^3 \times S^3 \times [0, 1]$ with $(x, 1)$ identified with $(f(x), 0)$ for each $x \in S^3 \times S^3$. Then $\pi_1(M) = Z$ and the universal cover \widetilde{M} is homotopy equivalent to $S^3 \times S^3$. Hence the Betti numbers of the loop space grow polynomially. On the other hand, the hyperbolic action of f on H_3 gives hope for exponential growth of the sum of the Betti numbers of $\Omega_T(p, q)$.

3.2. Stability of minimax levels (communicated by Nancy Hingston). Let M be a compact manifold with a Riemannian (or Finsler) metric g . Let G be a finitely generated abelian group. Given a nontrivial homology class $X \in H_*(\Lambda M; G)$, the critical (minimax) level of X is

$$\begin{aligned} cr X &= \inf \{a : X \in \text{Image } H_*(\Lambda_a M; G)\} \\ &= \inf_{x \in X} \sup_{\gamma \in \text{Image } x} \text{Length}(\gamma) \end{aligned}$$

Here $\Lambda_a M$ is the subset of the free loop space ΛM consisting of loops whose length is at most a . The second definition is the minimax definition: the singular chain x

ranges over all representatives of the homology class X , and γ over all the points in the image of x , which are loops in M .

Question 3.2.1 (Hingston). *Do there exist a metric on S^n and a homology class $X \in H_*(\Lambda M; \mathbb{Z})$ with*

$$0 < cr(mX) < cr(X)$$

for some $m \in \mathbb{N}$? The simplest case is already interesting: Can we find a metric on S^2 and $m \in \mathbb{N}$ so that $cr(mX) < cr(X)$, where X is a generator of $H_1(\Lambda S^2; \mathbb{Z})$?

Let us explain how this question is related to closed geodesics.

Given a metric g on M and a finitely generated abelian group G , the *global mean frequency* is defined as

$$(3.1) \quad \alpha_{g,G} = \lim_{\deg X \rightarrow \infty} \frac{\deg X}{cr X},$$

where the limit is taken over all nontrivial homology classes $X \in H_*(\Lambda S^n; G)$.

The Resonance Theorem from [Hin-Rad2013] says that if M is a sphere and G is a field, the limit (3.1) exists. It is clear in this case that $\alpha_{g,G}$ depends on the metric g . But does it really depend on the field G ? The degree of X does not depend on anything but X . But what about the critical level $cr X$? Does $cr X$ depend on the coefficients?

Let us note that for the spheres the nontrivial homology groups of the free loop space (with integer coefficients) are all \mathbb{Z} or $\mathbb{Z}_2 =: \mathbb{Z}/2\mathbb{Z}$. (For odd spheres they are all \mathbb{Z} .) Let us look at the case where $X \in H_k(\Lambda; \mathbb{Z}) = \mathbb{Z}$. For each $m \in \mathbb{N}$ there is a critical level

$$cr(mX) = \inf\{a : mX \in \text{Image } H_*(\Lambda_a M)\}.$$

If $j, m \in \mathbb{N}$, then clearly (since $\text{Image } H_*(\Lambda_a)$ is an additive subgroup of $H_*(\Lambda)$)

$$cr(jmX) \leq cr(mX).$$

But can there be strict inequality? Here is a little “example” to show how this would affect the global mean frequency : Suppose it were the case that there were real numbers $a < b < c < d$ with

$$cr(mX) = \begin{cases} d & \text{if } \gcd(m, 3) = \gcd(m, 7) = 1 \\ c & \text{if } 3|m \text{ but } \gcd(m, 7) = 1 \\ b & \text{if } 7|m \text{ but } \gcd(m, 3) = 1 \\ a & \text{if } 21|m \end{cases}$$

Conjecture 3.2.2 (Hingston).

$$\begin{aligned} \alpha_{g,G} &= a \text{ if } G = \mathbb{Q} \\ \alpha_{g,G} &= b \text{ if } G = \mathbb{Z}_3 \\ \alpha_{g,G} &= c \text{ if } G = \mathbb{Z}_7 \\ \alpha_{g,G} &= a \text{ if } G = \mathbb{Z}_p, p \neq 3, 7. \end{aligned}$$

4. CURVATURE CONDITIONS AND HYPERBOLICITY OF THE GEODESIC FLOW

Many results about geodesics and the geodesic flow assume that all sectional curvatures are negative or one of the following increasingly weaker properties:

- (1) all sectional curvatures are non positive,
- (2) no focal points,
- (3) no conjugate points.

These properties can be characterized by the behaviour of Jacobi fields:

- (0) Negative curvature: the length of any (non trivial) Jacobi field orthogonal to a geodesic is a strictly convex function.
- (1) Non positive curvature: the length of any Jacobi field is a convex function.
- (2) No focal points: the length of an initially vanishing Jacobi field is a non decreasing function along a geodesic ray.
- (3) No conjugate points: a (non trivial) Jacobi field can vanish at most once.

The no focal point property is equivalent to convexity of spheres in the universal cover. Most interesting results about manifolds with non positive curvature extend readily to manifolds with no focal points.

For a compact manifold, negative curvature implies that the geodesic flow is uniformly hyperbolic, in other words an Anosov flow. This means that there is a $D\phi_t$ -invariant splitting of the tangent bundle of the unit tangent bundle SM ,

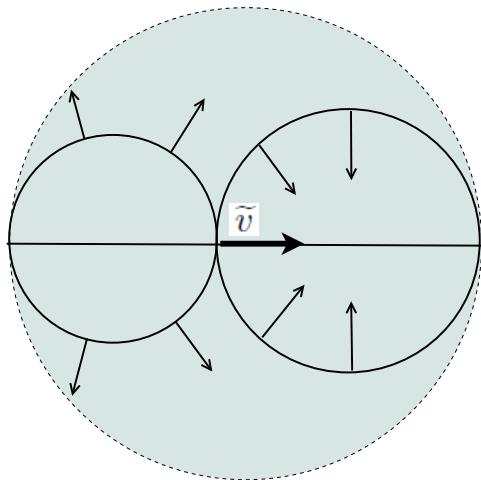
$$TSM = E^s \oplus E^0 \oplus E^u,$$

in which E^0 is the one dimensional subbundle tangent to the orbits of the geodesic flow, and there are constants $C \geq 1$ and $\lambda > 0$ such that for any $t \geq 0$ and any vectors $\xi \in E^s$ and $\eta \in E^u$ we have

$$(4.1) \quad \|D\phi_t(\xi)\| \leq Ce^{-\lambda t}\|\xi\| \quad \text{and} \quad \|D\phi_{-t}(\eta)\| \leq Ce^{-\lambda t}\|\eta\|.$$

(Here we have in mind the usual Sasaki metric; the same property would also hold for any equivalent metric with different constants C and λ .) This splitting is Hölder-continuous, but usually not smooth.

The bundles E^s and E^u for an Anosov geodesic flow are integrable; their integral foliations are usually denoted by W^s and W^u . The lifts to the universal cover of the leaves of W^s and W^u are closely related to horospheres. If \tilde{v} is the lift to \widetilde{SM} of $v \in SM$, the lifts to \widetilde{SM} of $W^s(v)$ and $W^u(v)$ are formed by the unit vectors that are normal to the appropriate horospheres orthogonal to \tilde{v} and are on the same side of the horosphere as \tilde{v} ; see the picture below.



4.1. Relations between these concepts. The notions introduced above are related as follows:

$$\begin{array}{ccc}
 \text{negative curvature} & \implies & \text{non positive curvature} & \implies & \text{no focal points} \\
 \Downarrow & & & & \Downarrow \\
 \text{Anosov geodesic flow} & \implies & & \implies & \text{no conjugate points}
 \end{array}$$

That Anosov geodesic flow implies no conjugate points is proved in part B of [Mañé1987].

The class of compact manifolds that support metrics with variable negative curvature is much larger than the class that support hyperbolic metrics (of constant negative curvature). The earliest examples of manifolds that support variable but not constant negative curvature were given by Mostow-Siu [Mos-Siu1980] and Gromov-Thurston [Gro-Thu1987]. Recent work of Ontaneda has vastly increased the supply of examples [Ontaneda2011, Ontaneda2014].

Conjecture 4.1.1 (Klingenberg). *If a closed manifold admits a metric with Anosov geodesic flow, then it admits a metric with negative sectional curvature.*

All known examples of metrics with Anosov geodesic flow are perturbations of metrics with negative curvature. It is not difficult to show using the uniformisation theorem and the Thurston geometrization theorem that conjecture is true in dimensions 2 and 3. Klingenberg [Klingenberg1974] showed that seven properties of Riemannian manifolds with negative curvature extend to those with Anosov geodesic flow. One of these is Preissman's theorem [Preissman1943] that the fundamental group of a manifold with negative sectional curvatures cannot contain a copy of $\mathbb{Z} \times \mathbb{Z}$.

Several examples of manifolds that admit metrics of non positive curvature but cannot support a metric of negative curvature (or with Anosov geodesic flow) can be found in the introduction to [Ba-Br-Eb1985]. These examples have a copy of $\mathbb{Z} \times \mathbb{Z}$ in their fundamental group. The simplest of them is due to Heintze who took two cusped hyperbolic 3-manifolds, cut off the cusps, glued the two pieces together

along their boundary tori and then smoothed out the metric to obtain a manifold with non positive curvature.

Question 4.1.2. *If (M, g) has no conjugate points, does M carry a metric with nonpositive sectional curvature?*

Again it is not difficult to use the uniformisation theorem to show that the answer is affirmative in dimension 2. But the problem is still open even in dimension 3. See [Cro-Sch1986], [Lebedeva2002] and [Iva-Kap2014] for results of the nature that the fundamental groups of manifolds with no conjugate points share properties with those of non positive curvature. Section 8 of [Iva-Kap2014] contains a number of open problems of which we mention:

Question 4.1.3. *Is the fundamental group of a closed manifold without conjugate points semihyperbolic?*

Semihyperbolicity is a condition introduced by Alonso and Bridson [Alo-Bri1995] to describe non-positive curvature in the large for an arbitrary metric space.

Question 4.1.4 (Hermann). *Is there an example of a geodesic flow that is partially hyperbolic?*

The geodesic flow is partially hyperbolic if there are a $D\phi_t$ -invariant splitting

$$TSM = E^s \oplus E^c \oplus E^u$$

and constants $C \geq 1$ and $\lambda > \mu > 0$ such that (4.1) holds and in addition for all t and all $\zeta \in E^c$ we have

$$C^{-1}e^{-\mu|t|}\|\zeta\| \leq \|D\phi_t(\zeta)\| \leq Ce^{\mu|t|}\|\zeta\|.$$

Anosov geodesic flows give a degenerate positive answer to the question. Genuine examples have been constructed by Carneiro and Pujals; see [Car-Puj2011] and [Car-Puj2013]. They deformed a higher rank locally symmetric space of non compact type.

5. NEGATIVE CURVATURE AND HYPERBOLICITY OF THE GEODESIC FLOW

5.1. Does the marked length spectrum determine the metric? Suppose (M, g) is a closed Riemannian manifold. The marked length spectrum of M is the function that assigns to each free homotopy class of loops the infimum of the length of loops in the class (i.e. the length of the shortest closed geodesic lying in this class).

Conjecture 5.1.1 ([Bur-Kat1985]). *Two metrics with negative curvature on a compact manifold must be isometric if they have the same marked length spectrum.*

Croke and Otal showed that this conjecture is true for metrics on surfaces [Otal1990, Croke1990]. Indeed it is enough to assume that the metrics have non-positive curvature. The problem is open in higher dimensions. Hamenstädt showed that the geodesic flows of the two metrics must be C^0 -conjugate, thereby reducing the problem to Conjecture 5.2.1 in the next subsection [Hamenstädt1991].

One obtains a natural interesting modification of this conjecture by replacing negative curvature by nonexistence of conjugate points. On the other hand some restriction on the metrics is necessary as the examples of Croke and Kleiner which we describe in the next subsection provide non-isometric metrics with the same

marked length spectrum. It is also necessary to consider the marked length spectrum rather than the length spectrum. Vignéras gave examples of non-isometric hyperbolic surfaces that have the same set of lengths for their closed geodesics [Vignéras1980].

One can ask a similar question for non-manifolds. Consider a 2-dimensional (metrical) simplicial complex such that every simplex is hyperbolic with geodesic edges and such that $CAT(-1)$ condition holds on every vertex. Assume in addition that every edge is contained in at least two simplices.

Question 5.1.2 (Schmidt). *Does the marked length spectrum determine such a metric (in the class of all locally $CAT(-1)$ metrics on this space, or in the class of all 2-dimensional (metrical) simplicial complexes with the above properties homeomorphic to the given complex)?*

If the simplicial complex is (topologically) a manifold, the answer is positive and is due to [Her-Pau1997]. One can also ask the question in higher dimensions. An easier, but still interesting version of the question is when we assume that every edge is contained in at least three simplices.

The questions above are closely related to the boundary rigidity problem, which we now recall. Given a compact manifold M with smooth boundary N , a Riemannian metric g on M induces a nonnegative real valued function d on $N \times N$ where $d(p, q)$ is the distance in (M, g) between p and q . We call (M, g) boundary rigid if a Riemannian manifold g' on M that induces the same function on $N \times N$ must be isometric to g' .

Question 5.1.3 ([Michel1981]). *What conditions on M , N and g imply boundary rigidity?*

One can modify this question by requiring that the other Riemannian metric g' in the definition of boundary rigid manifolds above also satisfies some additional assumption. Special cases of the last question were answered in [Croke1990, Cr-Da-Sh2000, Sha-Uhl2000, Bur-Iva2013].

One can also ask the boundary rigidity question for Finsler metrics; recent references with nontrivial results include [Coo-Del2010, Bur-Iva2010].

5.2. The conjugacy problem. Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) have C^k -conjugate geodesic flows if there is an invertible map $h : S_1^M \rightarrow S_2^M$ such that h and h^{-1} are C^k and $h \circ \phi_t^1 = \phi_t^2 \circ h$ for all t . Here ϕ_t^i is the geodesic flow for the metric g_i . A long standing conjecture is

Conjecture 5.2.1. Compact Riemannian manifolds with negative curvature must be isometric if they have C^0 -conjugate geodesic flows.

As with Conjecture 5.1.1 it may be possible to relax the hypothesis of negative curvature to no conjugate points, but some restriction on the metrics is required. Weinstein pointed out that all of the Zoll metrics on S^2 have conjugate geodesic flows. Using the explicit rotationally-symmetric examples of Zoll metrics (Tannery metrics in the terminology of [Besse1978, Chapter 4]) Croke and Kleiner constructed examples of different metrics on an arbitrary closed manifold such that their geodesic flows are conjugate and their marked length spectra coincide, see [Cro-Klei1994].

The conjecture is true in dimension two; indeed it is enough to assume that one of the manifolds has nonpositive curvature [Cr-Fa-Fe1992]. It follows from

the theorem of Besson-Courtois-Gallot discussed in the next subsection that the conjecture holds if one of the metrics is locally symmetric. Croke and Kleiner showed that C^1 conjugacy implies that the volumes of the manifolds are the same with no restrictions on the metrics and implies isometry of the metrics if one of them has a global parallel vector (as is the case in a Riemannian product) [Cro-Klei1994].

5.3. Entropy and locally symmetric spaces with negative curvature. Let h_{Liou} denote the entropy of the geodesic flow with respect to the Liouville measure, h_{top} its topological entropy and h_{vol} the volume entropy

$$h_{vol} = \lim_{r \rightarrow \infty} \frac{1}{r} \log \text{Vol } B_{\widetilde{M}}(p, r),$$

where $B_{\widetilde{M}}(p, r)$ is the ball of radius r around a point p in the universal cover \widetilde{M} . It is always true that $h_{vol} \leq h_{top}$ and they are equal if there are no conjugate points, in particular if the curvature is negative [Manning1979, Fre-Mañ1982]. By the variational principle for entropy $h_{Liou} \leq h_{top}$ for any metric; these entropies are equal for a locally symmetric metric.

Katok [Katok1982] showed that if g is an arbitrary metric and g_0 a metric of constant negative curvature on a surface of genus ≥ 2 such that $\text{area}(g) = \text{area}(g_0)$, then

$$h_{Liou}(g) \leq h_{Liou}(g_0) = h_{top}(g_0) \leq h_{top}(g)$$

and both inequalities are strict unless g also has constant negative curvature. He explicitly formulated the following influential and still open conjecture:

Conjecture 5.3.1 ([Katok1982]). $h_{top} = h_{Liou}$ for a metric of negative curvature on a compact manifold if and only if it is locally symmetric.

Katok's arguments extend to higher dimensions provided the two metrics are conformally equivalent. Flaminio showed that Conjecture 5.3.1 holds for deformations of constant curvature metrics [Flaminio1995].

Katok's paper implicitly raised the questions of whether in higher dimensions h_{top} is minimized and h_{Liou} maximized (among metrics of negative curvature of fixed volume on a given manifold) by the locally symmetric metrics. It is now known that the locally symmetric metrics are critical points for both entropies [Ka-Kn-We1991], but the locally symmetric spaces do not maximize h_{Liou} except in the two dimensional case [Flaminio1995]. The topological entropy, however, is minimized.

Gromov [Gromov1983] conjectured that if $f : (M, g) \rightarrow (M_0, g_0)$ is a continuous map of degree $d \neq 0$ between compact connected oriented n -dimensional Riemannian manifolds and M_0 has constant negative curvature, then

$$h_{vol}(g)^n \text{Vol}(M, g) \geq |d| h_{vol}(g_0)^n \text{Vol}(M_0, g_0)$$

with equality if and only if g also has constant negative curvature and f is homotopic to a d -sheeted covering. The quantity $h^n \text{Vol}$ has the advantage of being invariant under homothetic rescaling of the manifold; this obviates the assumption that the two manifolds have the same volume.

Theorem 5.3.2 ([Be-Co-Ga1995]). *The above conjecture of Gromov is true even when (M_0, g_0) is allowed to be a locally symmetric space of negative curvature.*

We refer the reader to Section 9 of [Be-Co-Ga1995] and the survey [Eberlein2001] for a list of the many corollaries of this remarkable theorem.

5.4. Regularity of the Anosov structure. An extensive list of results about the regularity of the Anosov splitting for the geodesic flow in negative curvature (and for other Anosov systems) can be found in the introduction to [Hasselblatt1994]. In dimension two, or if the curvature is 1/4-pinched, the splitting is always at least C^1 . In higher dimensions it is always Hölder continuous but typically not C^1 . It is natural to ask:

Question 5.4.1. *Does there exist a closed Riemannian manifold of negative curvature such that E^s and E^u are C^1 smooth, but the dimension is > 2 and the metric is not 1/4-pinched?*

Another motivation for this question is the question of Pollicott in the next subsection.

A program to construct open sets of metrics for whose geodesic flows have Anosov splitting that have low regularity of the Anosov splitting on large subsets of the unit tangent bundle is explained in section 4 of [Has-Wil1999]; see in particular Proposition 12. The idea is to perturb a suitable base example. This base example must have directions in its unstable manifolds with widely different expansion rates. Unfortunately there are no known examples of geodesic flows that are suitable bases for the perturbation argument; see the discussion at the end of section 4 in [Has-Wil1999]. Along all geodesics in complex hyperbolic space one has Lyapunov exponents of 1 and 2 corresponding to parallel families of planes with curvature -1 and -4 respectively.

Question 5.4.2. *Is there a metric of negative curvature for which the ratio of largest positive Lyapunov exponent to smallest positive Lyapunov exponent is greater than 2 for typical geodesics (not just exceptional closed geodesics)?*

The Anosov splitting is C^∞ only if and only if the metric is locally symmetric. This follows from [Be-Fo-La1992] and [Be-Co-Ga1995]. For surfaces Hurder and Katok [Hur-Kat1990] showed that the splitting must be C^∞ if one of the stable or unstable foliations is $C^{1+o(x \log |x|)}$. In higher dimensions it is expected that the splitting must be C^∞ if it is C^2 , but this question still seems to be open.

5.5. How many closed geodesics of length $\leq T$ exist? (Communicated by Pollicott). Let (M, g) be a closed Riemannian manifold with negative sectional curvatures. It is known [Margulis2004] that the number $N(T)$ of closed geodesics of length $\leq T$ grows asymptotically as e^{hT}/hT .

Conjecture 5.5.1. $N(T) = (1 + O(e^{-\varepsilon T})) \int_2^{e^{hT}} \frac{du}{\log u}$.

The conjecture is true for all metrics such that the stable and unstable bundles E^s and E^u are C^1 smooth, which is the case for surfaces and for 1/4 pinched metrics. Question 5.4.1 above asks whether there are other examples.

5.6. Infinitely many simple closed geodesics.

Question 5.6.1 (Miller [Miller2001], Reid). *Are there infinitely many simple closed geodesics in every hyperbolic three-manifold of finite volume?*

One can ask the same question for manifolds of variable negative curvature in any dimension. The answer is positive for surfaces, since in this case a free homotopy

class contains a simple closed geodesic if and only if it contains a simple closed curve. The answer is also positive for generic metrics of negative curvature.

Reid [Reid1993] has examples of (arithmetic) hyperbolic manifolds of finite volume in which every closed geodesic is simple.

6. NONPOSITIVE CURVATURE AND NON UNIFORM HYPERBOLICITY OF THE GEODESIC FLOW

Recall that the rank of a vector v in such a manifold is the dimension of the space of Jacobi fields along the geodesic γ_v that are covariantly constant. It is easily seen that rank is upper semi continuous. All vectors have rank ≥ 1 , since the velocity vector field is a covariantly constant Jacobi field. The rank of the manifold is the minimum rank of a vector. The set of vectors of minimum rank is obviously open and is known to be dense [Ballmann1982]. The definitions generalize the classical notion of rank for locally symmetric spaces of non compact type.

6.1. Ergodicity of geodesic flows.

Question 6.1.1. *Is the geodesic flow of a metric of non positive curvature on a closed surface of genus ≥ 2 ergodic with respect to the Liouville measure?*

It is known that the flow is ergodic on the set of geodesics passing through points where the curvature is negative [Pesin1977]. The complement of this set consists of vectors tangent to zero curvature geodesics, i.e. geodesics along which the Gaussian curvature is always zero. It is not known whether this set must have measure zero. There is an analogous question in higher dimensions: *is the geodesic flow of a closed rank one manifold of non positive curvature ergodic?* Here it is known that the flow is ergodic on the set of rank one vectors [Bal-Bri1982, Burns1983], but it is not known if the complementary set (of higher rank vectors) must have measure zero.

Several papers published in the 1980s (notably [Burns1983], [Bal-Bri1982], and [Ba-Br-Eb1985]) stated that this problem had been solved. These claims were based on an incorrect proof that the set of higher rank vectors must have measure zero.

The measure considered above is the Liouville measure. There is a (unique) measure of maximal entropy for the geodesic flow of a rank one manifold. It was constructed by Knieper, who proved that it is ergodic [Knieper1998].

6.2. Zero curvature geodesics and flat strips. Consider a closed surface of genus ≥ 2 .

Question 6.2.1. [Burns] *Is there a C^∞ (or at least C^k , $k \geq 3$) metric for which there exists a non closed geodesic along which the Gaussian curvature is everywhere zero?*

A negative answer to this question would give a positive answer to Question 6.1.1.

If we assume that the curvature is only C^0 , then a metric with such a geodesic can be constructed. Each end of the geodesic is asymptotic to a closed geodesic. But a geodesic along which the curvature is zero cannot spiral into a closed geodesic if the curvature is C^2 ; see [Ruggiero1998]. Wu [Wu2013] has recently given a negative answer to the question under the assumption that the subset of the surface where the curvature is negative has finitely many components; he also shows under this hypothesis that only finitely many free homotopy classes can contain zero curvature closed geodesics.

A flat strip is a totally geodesic isometric immersion of the Riemannian product of \mathbb{R} with an interval. Zero curvature geodesics can be viewed as flat strips with infinitesimal width. It is known that all flat strips (with positive width) must close up in the \mathbb{R} -direction; in other words they are really immersions of the product of S^1 with an interval. Furthermore for each $\delta > 0$ there are only finitely many flat strips with width $\geq \delta$. Proofs can be found in a preprint of Cao and Xavier [Cao-Xav]. The main idea is that if two flat strips of width δ cross each other at a very shallow angle, then their intersection contains a long rectangle with width close to 2δ .

It is possible that there are only finitely many flat strips. This is true when the set where the curvature is negative has finitely many components [Wu2013], but the only effort to prove it in general [Rodriguez Hertz2003] was unsuccessful.

Remark. Question 6.2.1 still makes sense even if the curvature of the manifold is not restricted to be non positive. It might generalize also to higher dimensions. What can be said about a geodesic along which all sectional curvatures are 0? Or less stringently the sectional curvatures of planes containing the tangent vector?

6.3. Flats in rank one manifolds with nonpositive curvature. A flat is a flat strip of infinite width, in other words a totally geodesic isometric immersion of the Euclidean plane. Eberlein asked whether a compact rank one manifold that contains a flat must contain a closed flat, i.e. a totally geodesic isometric immersion of a flat torus. Bangert and Schroeder gave an affirmative answer for real analytic metrics [Ban-Sch1991]. The problem is open for C^∞ metrics.

6.4. Besson-Courtois-Gallot. Does their rigidity theorem from [Be-Co-Ga1995], Theorem 5.3.2 in this paper, generalize to higher rank symmetric spaces of non compact type?

Connell and Farb [Con-Far2003a, Con-Far2003b] extended the barycenter method, which plays a vital role in [Be-Co-Ga1995], and proved that the theorem holds for a product in which each factor is a symmetric space with negative curvature and dimension ≥ 3 . We also refer the reader to their extensive survey in [Con-Far2003c].

A recent preprint of Merlin [Merlin2014] uses a calibration argument to show that $h_{vol}^4 \text{Vol}$ is minimized by the locally symmetric metric on a compact quotient of the product of two hyperbolic planes.

6.5. Closed geodesics. Consider a closed surface of genus ≥ 2 with a metric of nonpositive curvature. Let $N(T)$ be the number of free homotopy classes containing closed geodesics along which the Gaussian curvature is everywhere zero.

Question 6.5.1 (Knieper). *Suppose $N(T)$ grows subexponentially, i.e.*

$$\lim_{T \rightarrow \infty} \frac{\log N(T)}{T} = 0.$$

Is there a quadratic upper bound on $N(T)$, i.e., does there exist $C > 0$ such that $N(T) \leq C \cdot T^2$?

The answer is positive for the torus. One can also study this problem in the Finsler category.

7. MANIFOLDS WITHOUT CONJUGATE POINTS (RIGIDITY CONJECTURES)

There are many rigidity results for manifolds with non positive curvature that might extend to manifolds with no conjugate points. One still has the basic setting of a universal cover homeomorphic to \mathbb{R}^n in which any two points are joined by a unique geodesic. However the convexity of the length of Jacobi fields, which is the basis of many arguments used in non positive curvature, is no longer available.

Two major results of this nature are:

Theorem 7.0.2. *A Riemannian metric with no conjugate points on a torus is flat.*

Theorem 7.0.3. *Let g be a complete Riemannian metric without conjugate points on the plane \mathbb{R}^2 . Then for every point p*

$$\liminf_{r \rightarrow \infty} \frac{\text{area } B(p, r)}{\pi r^2} \geq 1,$$

with equality if and only if g is flat.

The two dimensional case of Theorem 7.0.2 was proved by E. Hopf [Hopf1948] and the general case by Burago and Ivanov [Bur-Iva1994]. The Lorentzian analogue of Theorem 7.0.2 is false; two dimensional counterexamples are constructed in [Bav-Mou2013].

Theorem 7.0.3 is a recent result of Bangert and Emmerich [Ban-Emm2013], which greatly improved on earlier results in [Bur-Kni1991] and [Ban-Emm2011]. Both results are easy for manifolds with non positive curvature but require subtle arguments in the context of no conjugate points. Bangert and Emmerich's work was motivated by the following conjecture, which they prove using their theorem:

Conjecture 7.0.4 (Bangert, Burns and Knieper). *Consider a complete Riemannian metric without conjugate points on the cylinder $\mathbb{R} \times S^1$. Assume that the ends spread sublinearly, i.e.*

$$\lim_{d(p, p_0) \rightarrow \infty} \frac{l(p)}{\text{dist}(p, p_0)} = 0,$$

where $l(p)$ is the length of the shortest geodesic loop with the based at p . Then the metric is flat.

7.1. Divergence of geodesics. Let α and β be two geodesics in a complete simply connected Riemannian manifold without conjugate points.

Question 7.1.1. *Suppose $\alpha(0) = \beta(0)$. Does $\text{dist}(\alpha(t), \beta(t)) \rightarrow \infty$ as $t \rightarrow \infty$?*

The answer is positive in dimension 2 in [Green1954]. The question is open in higher dimensions; the proof in [Green1956] is incorrect.

A closely related question is:

Question 7.1.2. *Suppose $\text{dist}(\alpha(t), \beta(t)) \rightarrow 0$ as $t \rightarrow -\infty$. Does $\text{dist}(\alpha(t), \beta(t)) \rightarrow \infty$ as $t \rightarrow \infty$?*

Without positive answers to these questions there would seem to be little hope for a satisfactory analogue of the sphere at infinity, which plays a prominent role in the case of non positive curvature.

7.2. Parallel postulate questions.

The flat strip theorem is another basic tool in the study of manifolds with non positive curvature. It states that “parallel geodesics” must bound a flat strip. More precisely if α and β are two geodesics in a simply connected manifold with non positive curvature such that $\text{dist}(\alpha(t), \beta(t))$ is bounded for all $t \in \mathbb{R}$, then the two geodesics are the edges of a totally geodesic isometric immersion of the Riemannian product of \mathbb{R} with an interval.

The flat strip theorem fails for manifolds with no conjugate points (although it does extend fairly easily to manifolds without focal points). Counter examples have been given in [Burns1992] and by Kleiner (unpublished). Kleiner’s example has a copy of $\mathbb{Z} \times \mathbb{Z}$ in its fundamental group, but does not contain a corresponding flat torus. He perturbs the Heintze example (Example 4 in [Ba-Br-Eb1985]), which we described in subsection 4.1. The following basic question appears to be open in general (although Proposition 4 in [Eschenburg1977] suggests an affirmative answer under some extra hypotheses).

Question 7.2.1. *Must homotopic closed geodesics in a manifold with no conjugate point have the same length?*

One can still hope for a version of the theorem in manifolds with no conjugate points in which there is a large enough family of “parallel” geodesics. One would then hope to find a totally geodesic isometric immersion of the Euclidean plane. Rigidity might hold in the large, even though the examples above show that it breaks down locally.

The simplest question of this type asks if a Riemannian plane satisfying Euclid’s 5th postulate must be flat. Consider the plane \mathbb{R}^2 with a complete Riemannian metric. Assume that for every geodesic and for every point not on the geodesic there exists precisely one nontrivial geodesic that passes through the point and does not intersect the geodesic. This assumption implies that the metric has no conjugate points.

Question 7.2.2 ([Bur-Kni1991]). *Must a metric satisfying this version of the parallel postulate be flat?*

The question looks like a question in the synthetic geometry, but it is not, since we do not require a priori that the other axioms of the Euclidean geometry are fulfilled (for example the congruence axioms).

7.2.1. Higher rank rigidity.

Conjecture 7.2.3 (Spatzier). *Let (M, g) be a closed symmetric space of noncompact type and of higher rank. Then the only metrics on M with no conjugate points are homothetic rescalings of g .*

7.3. Is $M \times S$ with no conjugate points a direct product? Consider the product of a closed surface M^2 of genus ≥ 2 with the circle S . Let g be a Riemannian metric on $M \times S$ with no conjugate points.

Conjecture 7.3.1 (Burago-Kleiner). *The metric on the \mathbb{Z} -cover corresponding to S^1 factor is a direct product: $(M \times \mathbb{R}, g) = (M, g_1) \times (\mathbb{R}, dt^2)$.*

One can of course make more general conjectures, e.g. about metrics without conjugate points on products, or on manifolds which admit nonpositively curved

metrics with higher rank, but the above conjecture with $M \times S^1$ seems to be the easiest one (and is probably still hard to prove).

7.4. Magnetic geodesics without conjugate points. One can generalize the notion “conjugate points” for magnetic geodesics (even for arbitrary natural Hamiltonian systems.)

Question 7.4.1 (Paternain). *Consider a magnetic flow on a closed surface of genus ≥ 2 and an energy level such that the topological entropy vanishes. Suppose that the magnetic geodesics on this level do not have conjugate points. Is the system locally symmetric in the sense that the metric has constant curvature and the magnetic form is a constant multiple of the volume form?*

Many of the earlier questions in this section can also be asked about magnetic geodesics lying on a certain (possibly sufficiently high) energy level. Let us mention [Bialy2000], where a natural analog of Theorem 7.0.2 was proved under the assumption that the metric is conformally flat, and it was conjectured that this assumption is not essential.

8. UNRESTRICTED CURVATURE

8.1. Ergodic geodesic flows. *Does every closed manifold (with dimension ≥ 2) admit a metric (Riemannian or Finsler) with ergodic geodesic flow?*

This is known for surfaces [Donnay1988II], for 3-manifolds [Katok1994], for product manifolds in which the factors have dimension ≤ 3 [Bur-Ger1994], and for spheres [Bur-Ged]. Donnay and Pugh have shown that any embedded surface can be perturbed, in the C^0 topology to an embedded surface whose geodesic flow is ergodic [Don-Pugh2004]. These constructions all make essential use of the focusing caps introduced by Donnay in [Donnay1988I].

The general problem is still wide open despite some reports of its solution (p. 87 of [Berger2000] and Section 10.9 of [Berger2003]).

8.2. Positive curvature.

Question 8.2.1. *Is there a Riemannian metric on a closed manifold with everywhere positive sectional curvatures and ergodic geodesic flow?*

Metrics close to the standard metric on S^2 would be especially interesting.

8.3. The measure of transitive and recurrent sets. Let (M^2, g) be a closed surface with ergodic geodesic flow. Then every tangent vector lies in one of the following sets:

$$\begin{aligned} T_b &:= \{v \in SM \mid \text{any lift of } \gamma_v \text{ stays in a bounded subset of } \widetilde{M}\}, \\ T_p &:= \{v \in SM \mid \text{any lift of } \gamma_v \text{ is unbounded, and approaches infinity properly}\}, \\ T_i &:= TM \setminus (T_b \cup T_p). \end{aligned}$$

All three sets are measurable and invariant under the geodesic flow. By ergodicity one of them has full measure.

Question 8.3.1 (Schmidt). *Which of these sets has full measure?*

It is expected that the answer depends on the genus of the surface: on the torus the set T_i has full measure, and on the surfaces of higher genus T_p has full measure.

In this context it is natural to consider only minimizing geodesics: if one shows that the set of minimizing geodesics has nonzero measure, then it has full measure by ergodicity of the geodesic flow. However, there exist examples of metrics on the torus such that the set of minimising geodesics has small Hausdorff dimension.

The question seems to make sense in any dimension.

8.4. Generic metrics. Consider a smooth closed manifold M and denote by \mathcal{G} the space of all smooth Riemannian metrics on M . One can equip this space with a natural C^k -topology: locally, in a coordinate chart it is induced by a C^k -norm of the metric g viewed as an $n \times n$ matrix in this coordinate chart. More precisely, let us consider a finite number U_1, \dots, U_m of charts that cover the manifold. We say that two metrics g and g' are ϵ -close if for any coordinate chart U_s and for all $i, j \leq n$ the functions g_{ij} and g'_{ij} (the (i, j) -components of the metrics in the coordinate chart U_s) are ϵ -closed in the standard C^k -norm, i.e., the difference of the values of these functions and all their partial derivatives up to the order k is less than ϵ). The notion of ϵ -closeness induces a notion of a ball in the space of the metrics and also a topology in the space of the metrics: a ball with center g is the set of g' that are ϵ -close to g . As usual, we call a subset of \mathcal{G} open if for any point of the subset a ball around the point is contained in the subset.

It is an easy exercise to show that the topology does not depend on the choice of the charts U_1, \dots, U_m used to define it.

Question 8.4.1. Do the metrics with positive entropy form a dense subset of \mathcal{G} ?

This question can be asked for any k . For $k = 2$ it was recently positively answered in all dimensions in [Contreras2010]; see also [Con-Pat2002]. For $k = \infty$ and dimension 2, it was answered positively in [Kni-Wei2002].

Question 8.4.2. Assuming $n = \dim(M) \geq 3$, do the metrics for which the geodesic flow is transitive form a dense subset of \mathcal{G} ?

Recall that a geodesic flow is transitive, if there exists a geodesic γ with $|\dot{\gamma}| = 1$ such that the set of its tangent vectors is dense on SM .

If $n = \dim(M) = 2$, the answer is negative: if we take a metric with integrable geodesic flow such that certain Liouville tori are irrational, any small perturbation of the metric has nontransitive geodesic flow by the KAM theory.

This question is also closely related to the famous Arnold diffusion conjecture.

Question 8.4.3. *Is there a dense subset of metrics in \mathcal{G} for which the tangent vectors to the closed geodesics are dense in SM ?*

Note that it is relatively easy to construct, see e.g. [Weinstein1970], a metric on any manifold such that a certain open subset of SM contains no vectors tangent to a closed geodesic.

8.5. Density in the manifold. In the previous section we discussed the unit tangent bundle SM . In this section we ask similar questions about M itself, and we will not assume that the metrics are generic.

Question 8.5.1. *Is the union of the closed geodesics always dense in the manifold?*

Question 8.5.2. *Does every metric on a compact surface of positive genus have a geodesic that is dense in the surface?*

8.6. Gaidukov in higher dimensions. We consider a Riemannian metric on an oriented closed surface M of genus ≥ 1 . Let Γ be a nontrivial free homotopy class and p a point in M . A theorem of Gaidukov [Gaidukov1966] says that there exist a closed geodesic $\gamma : \mathbb{R} \rightarrow M$ in Γ and a ray $\beta : [0, \infty) \rightarrow M$ with $\beta(0) = p$ such that $\text{dist}(\beta(t), \gamma(t)) \rightarrow 0$ as $t \rightarrow \infty$. As explained in [Bia-Pol1986], Gajdukov's results follow easily from the classical results of [Morse1924] and [Hedlund1932].

Problem 8.6.1 (Schmidt). *Generalize this statement to higher dimensions.*

Gaidukov's proof is profoundly two-dimensional and cannot be generalized. But Mather's work on minimizing orbits of Lagrangian systems (see eg. [Mather1991] or [Con-Itu1999]) might offer an approach. Instead of a single closed geodesic one should consider a minimizing set, namely the support of a minimal measure.

8.7. Systolic and diastolic inequalities for surfaces (communicated by Guth, Rotman and Sabourau). Let (M, g) be a compact Riemannian surface. The inequalities in question compare the length of certain short closed geodesics with the square root of the area of (M, g) . Recall that the systole $\text{sys}(M, g)$ is the least length of a non trivial closed geodesic. Two other geometrically meaningful constants can be defined by the following minimax procedure:

$$L(M, g) = \inf_f \max_{t \in \mathbb{R}} F[f^{-1}(t)],$$

in which the infimum is taken over all (Morse) functions $f : M \rightarrow \mathbb{R}$ and the functional F is either (a) the total length of $f^{-1}(t)$ or (b) the length of its longest component. In both cases $L(M, g)$ is realized as the length of a certain union of closed geodesics. In case (a), $L(M, g)$ is one definition of the diastole $\text{dias}(M, g)$ of the Riemannian surface (at least two different notions of diastole have been studied; see [Bal-Sab2010]).

By a result originally due to [Croke1988] and later improved in [Nab-Rot2002], [Sabourau2004] and [Rotman2006], for every Riemannian metric g on the sphere S^2 one has

$$\text{sys}(S^2, g) \leq \sqrt{32} \sqrt{\text{area}(S^2, g)}.$$

Actually, in [Croke1988] it was suggested that the constant $\sqrt{32}$ in the above inequality can be replaced by $\sqrt{2\sqrt{3}}$. The following example due to [Croke1988] shows that one can not go below $\sqrt{2\sqrt{3}}$: take two congruent equilateral triangles and glue them along their boundaries. The example is not smooth and suggests the study of systolic inequalities on the space of smooth metrics with conical singularities. On the space of such metrics, let us consider the function

$$\sigma(g) = \frac{\text{area}(S^2, g)}{\text{sys}(S^2, g)^2}.$$

This function has many nice properties; for example it is Lipschitz with respect to the appropriate distance on the space of metrics. By [Croke1988], the function has a positive minimum, and the natural conjecture is that the minimum is attained on the sphere constructed from two equilateral triangles as described above. But other critical points of this functions are also interesting: for example the round metric of the sphere is a critical point because of the huge set of Zoll metrics having the same value of σ as the round metric, see [Balacheff2006].

Question 8.7.1 (Babenko). *Does there exist, in the space of metrics with conical singularities, a (continuous or smooth) family of metrics g_t such that g_0 is the round metric of the sphere, g_1 is the metric from the example above, and $\sigma(g_t)$ is a decreasing function of t .*

A simpler version of this question is

Question 8.7.2 (Babenko). *Does every Riemannian metric on S^2 with area 4π that is close enough to the round metric have a closed geodesic with length $\leq 2\pi$?*

For surfaces of higher genus Gromov proved the existence of a constant C such that

$$\text{sys}(M, g) \leq C \log(\text{genus}(M)) \sqrt{\text{area}(M, g)};$$

the dependence on the genus of M in this inequality is sharp. For the diastole defined by (a), Balacheff and Sabourau showed in [Bal-Sab2010] that there is a constant C such that

$$\text{dias}(M, g) \leq C \sqrt{\text{genus}(M)} \sqrt{\text{area}(M, g)};$$

the dependence of this inequality on the genus is again optimal.

Question 8.7.3 (Guth). *Is there a constant C such that the invariant defined by (b) above is bounded from above by $C \sqrt{\text{area}(M, g)}$?*

An affirmative answer to Guth's question would mean that the three quantities under consideration all depend on the genus in different ways, and are therefore measuring different features of the surface. A positive answer would also show that one can always find a pants decomposition of a closed Riemannian surface of genus g into $3g - 3$ disjoint closed geodesics of length at most $C \sqrt{\text{area}(M)}$. This would precisely give the optimal Bers' constant for a genus g surface. Even for hyperbolic surfaces, this question is still open (see [Buser1992] for partial results).

The question above also makes sense in higher dimensions, but the relation with closed geodesics is not clear in this context. In this case $f^{-1}(t)$ would be an $(n-1)$ -complex, where n is the dimension of the manifold, and F would measure its total volume or the volume of its largest component.

8.8. Questions related to the systole in higher dimensions.

Question 8.8.1 (Question 4.11 of [Gromov2001]). *Does a Riemannian metric on a real projective space with the same volume as the canonical metric have a closed geodesic with length $\leq \pi$?*

In dimension two, an affirmative answer is in [Gromov2001, Proposition 4.10].

Question 8.8.2 (Alvarez Paiva). *Can there be Riemannian metrics on S^3 or $S^2 \times S^1$ all of whose closed geodesics are long? More specifically, does every metric on these spaces have a closed geodesic with length $\leq 10^{24}$ if the volume is 1?*

8.9. Metrics such that all geodesics are closed. This is a classical topic — the first examples go back at least to [Zoll1903]; see [Besse1978] for details. The book [Besse1978] is still up to date, and many problems/questions listed in it (in particular in Chapter 0 §D) are still open. As the most interesting question from their list we suggest:

Question 8.9.1. *Let M be a closed manifold not covered by a sphere. Let g be a metric on M for which all geodesics are closed. Is (M, g) a CROSS (=compact rank one symmetric) manifold (with the standard metric)?*

Of course, variants of this question can be asked about Finsler and Lorentzian metrics. In the Finsler setting, one may ask for example to describe all Finsler metrics on $CP(n)$ such that their geodesics are geodesics of the standard (Fubini-Study) metric on $CP(n)$. In the Lorentzian setting, one can ask to construct all manifolds such that all light-like geodesics are closed, see e.g. [Mou-Suh2013, Suhr2013b].

9. LORENTZIAN METRICS AND METRICS OF ARBITRARY SIGNATURE.

9.1. Closed geodesics. Most of the questions we asked about the Riemannian and Finsler metric can be modified such that they are also interesting in the semi-Riemannian metrics (i.e. of arbitrary signature) and in particular when the signature is Lorentzian. It appears though that the answers in the Lorentzian case are sometimes very different from those in the Riemannian case. Many methods that were effectively used in the Riemannian case, for example the variational methods, do not work in the case of general signature.

Let us consider as an example the question analogous to the one we considered in Section 2: how many geometrically different closed geodesics must there be for a Lorentzian metric on a closed manifold.

First let us note that there are two possible natural notions of closed geodesic in the Lorentzian setting: one may define a closed geodesic as an embedding $\gamma : S^1 \rightarrow M$ such that $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, or as a curve $\gamma : S^1 \rightarrow M$ that can be locally reparameterized in such a way that it satisfies the equation $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

In the Riemannian case, these definitions are essentially equivalent, since this reparameterization can always be made “global”. If the signature is indefinite, it is easy to construct examples of an embedding $\gamma : S^1 \rightarrow M$ such that locally γ can be reparameterized so that it becomes an affinely-parameterized geodesic, but globally such reparameterization is impossible: the velocity vector of the geodesic after returning to the same point is proportional but not equal to the initial velocity vector. Of course, this is possible only if the velocity vector is light-like.

We will follow most publications and define a closed geodesic as an embedding $\gamma : S^1 \rightarrow M$ such that $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

To the best of our knowledge, the existence of a closed geodesic on a Lorentzian manifold is a quite complicated problem and nothing is known in dimensions ≥ 3 . In dimension 2, a closed orientable manifold with a metric of signature $(+, -)$ is homeomorphic to the torus. By [Suhr2013a], every Lorentzian 2-dimensional torus has at least two simple closed geodesics one of which is definite, i.e. timelike or spacelike. Moreover, explicit examples show the optimality of this claim.

We therefore ask the following

Question 9.1.1. *Does every closed Lorentzian manifold have at least one closed geodesic?*

One of course can ask this question about manifolds of arbitrary signature and also about complete (note that in the Lorentzian setting there are many different nonequivalent notions of completeness) manifolds of finite volume.

See the introduction to the paper [Fl-Ja-Pi2011] for a list of known results about the existence of closed time-like geodesics under additional assumptions.

9.2. Light-like geodesics of semi-Riemannian metrics. Let M be a closed manifold with semi-riemannian metric g of indefinite signature.

Question 9.2.1. *Can there exist a complete semi-Riemannian metric g and a nontrivial 1-form η on a closed manifold such that for every light-like geodesic $\gamma(t)$ the function $\eta(\dot{\gamma}(t))$ grows linearly in both directions: i.e. for every geodesic there exist $C_1 \neq 0, C_2$ such that $\eta(\dot{\gamma}(t)) = C_1 \cdot t + C_2$?*

A negative answer to this question would give an easy proof of the semi-Riemannian version of the projective Lichnerowicz-Obata conjecture:

Let a connected Lie group G act on a complete manifold $(M^{n \geq 2}, g)$ by projective transformations (diffeomorphisms take geodesics to geodesics without necessarily preserving the parametrization). Then G acts by isometries, or g has constant sectional curvature.

If g is Riemannian, the conjecture was proved in [Matveev2005, Matveev2007]. The proof is complicated, and does not generalize to the semi-Riemannian setting in dimensions ≥ 3 (in dimension 2, under the assumption that the manifold is closed, the conjecture was proved in [Matveev2012b]). Fortunately, in the semi-Riemannian case the following argument gives a proof for closed manifolds provided that the answer to the question above is positive.

It is known (see, for example, [Matveev2007]) that a 1-form η_i generates a one-parameter group of projective transformations, if and only if

$$\eta_{i,jk} + \eta_{j,ik} - \frac{2}{(n+1)} \eta^{\ell}_{,\ell k} g_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}$$

(for a certain 1-form λ_i). We take a light line geodesic $\gamma(t)$, multiply the above equation by $\dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^k$ (and sum with respect to repeating indexes) at every point $\gamma(t)$ of the geodesic. The terms with the metric g disappear since $g_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0$, so we obtain the equation $\frac{d^2}{dt^2} \eta(\dot{\gamma}(t)) = 0$ implying that $\eta(\dot{\gamma}(t)) = t \cdot C_1 + C_2$. A negative answer to the question above implies that $C_1 = 0$. Hence $\eta(\dot{\gamma}(t))$ is constant, which in turn implies that $\eta_{i,j} + \eta_{j,i}$ is proportional to g_{ij} . Then the covector field η_i generates a one-parameter group of conformal transformations. Finally, the proof of the conjecture follows from a classical observation of H. Weyl [Weyl1921] that every transformation that is projective and conformal is a homothety.

A version of the question above is *whether, for a complete semi-Riemannian metric on a closed manifold M , the tangent vector of almost every geodesic remains in a bounded set of TM* . A positive answer on this question immediately implies that the answer to the initial question is negative, thereby proving the projective Lichnerowicz-Obata conjecture on closed manifolds.

9.3. Completeness of closed manifolds of arbitrary signature (communicated by H. Baum). It is well-known that a closed Riemannian manifold is geodesically complete, in the sense that every geodesic $\gamma : (a, b) \rightarrow M$ can be extended to a geodesic $\tilde{\gamma} : \mathbb{R} \rightarrow M$ such that $\tilde{\gamma}|_{[a,b]} = \gamma$. It is also well known that for any indefinite signature there exist metrics on closed manifolds that are not geodesically complete.

Question 9.3.1. *What geometric assumptions imply that a metric (possibly, of a fixed signature) on a closed manifold is geodesically complete?*

We of course are interested in geometric assumptions that are easy to check or which are fulfilled for many interesting metrics.

Let us mention a few classical results. For compact homogeneous manifolds, geodesic completeness was established in [Marsden1972]. For compact Lorentz manifolds of constant curvature geodesic completeness was proved in [Carrière1989] (flat case) and [Klingler1996] (general case).

We refer to the paper [Sanchez2013] for a list of interesting results on this topic and for a list of open questions from which we repeat here only:

Question 9.3.2 ([Sanchez2013]). *Assume that a compact Lorentzian manifold is globally conformal to a manifold of constant curvature. Must it be geodesically complete?*

Note also that in the noncompact case homogeneous manifolds of indefinite signature are not necessarily geodesically complete; see for example [Sanchez2013, Example 2 in §4]. It is interesting to understand whether completeness of a homogeneous manifold can follow from algebraic properties of the isometry group.

10. INTEGRABILITY AND ERGODICITY OF GEODESIC FLOWS ON SURFACES OF HIGHER GENUS

10.1. Metrics with integrable geodesic flow on surfaces of genus ≥ 2 .

Question 10.1.1 (Bangert). *Does there exist a Riemannian metric on a closed surface of genus ≥ 2 whose geodesic flow is completely integrable?*

The answer may depend on what we understand by “completely integrable”: whether the integral is functionally independent of the Hamiltonian on an open everywhere dense subset, or we additionally assume that the subset has full measure.

Mañé showed that a Hamiltonian flow on surfaces is generically Anosov or has zero Liouville exponents [Mañé1996]. This suggests that an easier version of the above question would be the following

Question 10.1.2 (Paternain). *Is there a Riemannian metric on a closed surface of genus ≥ 2 whose geodesic flow has zero Liouville entropy?*

Bangert and Paternain have outlined a nonconstructive proof that there are Finsler metrics with this property; finding a Riemannian metric is certainly harder.

A special case of the Finsler version of the question above would be:

Question 10.1.3. *Let F_1, F_2 be Finsler metrics on a closed surface of genus ≥ 2 . Assume every (unparameterized) F_1 -geodesic is an F_2 -geodesic. Must F_1 be obtained from F_2 by multiplication by a constant and adding a closed form?*

One can also ask this question for arbitrary closed manifolds that can carry a hyperbolic metric (if F_1, F_2 are Riemannian, the answer is affirmative [Matveev2003]).

The previous question is related to the other questions in this section because of the following observation from [Mat-Top1998]: one can use the second metric to construct an integral of the geodesic flow of the first one. If both metrics are Riemannian, the integral is quadratic in velocities and the affirmative answer follows from [Kolokoltsov1983]; see [Mat-Top2000].

Question 10.1.4. *Does there exist a nonriemannian Finsler metric satisfying the Landsberg condition on a surface of genus ≥ 2 ?*

The Landsberg condition is defined in [Bao2007]. It implies the existence of an integral for the geodesic flow of the metric, which is the relation of this question with the present section. See [Gom-Rug2013] for a negative answer to the question assuming nonexistence of conjugate points.

10.2. Integrable geodesic flows with integrals of higher degree.

Conjecture 10.2.1 ([Bo-Ko-Fo1995]). *If the geodesic flow of a Riemannian metric on the torus T^2 admits an integral that is polynomial of degree 3 in the velocities, then the metric admits a Killing vector field.*

A Killing vector field V allows us to construct an integral

$$I : TM \rightarrow \mathbb{R}, I(\xi) := g(V, \xi)$$

that is evidently linear in velocities; its third power is then an integral cubic in velocities.

The motivation to study metrics admitting integrals polynomial in velocities comes from the following observation (which dates back at least to Darboux and Whittaker): if the geodesic flow admits an integral that is analytic in velocities, then each component of this integral that is homogeneous in velocities is also an integral. The natural idea is then to study the integrals that are polynomial in velocities of low degree.

By the result of Kolokoltsov [Kolokoltsov1983], no metric on a surface of genus ≥ 2 admits an integral that is polynomial in velocities and is functionally independent of the energy integral. The state of the art if the surface is the sphere or the torus can be explained by the following table:

	Sphere S^2	Torus T^2
Degree 1	All is known	All is known
Degree 2	All is known	All is known
Degree 3	Series of examples	Partial negative results
Degree 4	Series of examples	Partial negative results
Degree ≥ 5	Nothing is known	Nothing is known

In the table, “Degree” means the smallest degree of a nontrivial integral polynomial in velocities. “All is known” means that there exists an effective description and classification (which can be found in [Bo-Ma-Fo1998]).

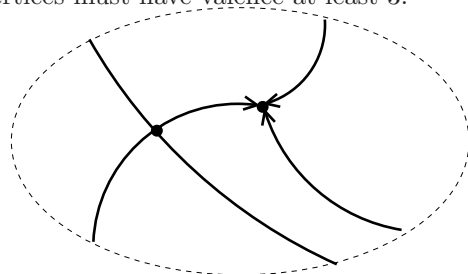
A simpler version of the question is when we replace the geodesic flow in the question above by a Lagrangian system with the Lagrangian of the form $L(x, \xi) := K + U = \sum g_{ij} \xi^i \xi^j + U(x)$. In this case we assume that the integral is a sum of polynomials of degrees 3 and 1 in velocities. The “partial negative results” in the table above correspond to this case; moreover, in most cases it is assumed that the metric g_{ij} is flat, see for example [Bialy1987, Mironov2010, De-Ko-Tr2012] (and [Bialy2010] for results that do not require this assumption).

Similar conjectures could be posed for integrals of every degree. If the degree d is odd, the conjecture is that the existence of an integral that is polynomial of the degree d in velocities implies the existence of a Killing vector field. If the degree d is even, the conjecture is that the existence of an integral that is polynomial of the degree d in velocities implies the existence of an integral quadratic in velocities and not proportional to the energy integral.

10.3. Metrics such that one can explicitly find all geodesics. Geodesics of a metric are solutions of a nonlinear ordinary differential equation $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ which can not be *explicitly* solved in most cases. There are only a few examples of 2-dimensional metrics for which one can explicitly find *all* geodesics using elementary functions: they are metrics of constant curvature and Darboux-superintegrable metrics (see [Br-Ma-Ma2008] for definition). An interesting problem is to construct other examples of metrics such that all geodesics are explicitly known. A related problem (asked recently in [Tao2010]) is to construct metrics with an explicitly given distance function.

11. STATIONARY NETS (COMMUNICATED BY ROTMAN).

A graph G in a Riemannian surface (S, g) is called a *stationary net*, if every edge is a geodesic and if at every vertex the sum of diverging unit vectors is zero. Vertices must have valence at least 3.

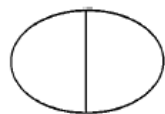


A fragment of a stationary net

11.1. Stationary Θ -nets. A Θ -graph is a graph that looks like the Greek letter “Theta”: two vertices connected by three edges, see the picture below.

Question 11.1.1 ([Has-Mor1996]). *Does every metric on S^2 admit a stationary Θ -graph?*

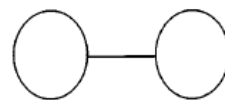
Partial results in this direction were obtained in [Has-Mor1996]: they showed that on every sphere of positive curvature there exists a stationary Θ -net, or a stationary eight curve net, or a stationary eyeglasses net.



Θ -graph



eight curve



eyeglasses

11.2. Density of stationary nets and of closed geodesics. We call a stationary net nontrivial if it has at least one vertex of valency ≥ 3 , in other words if it is not a disjoint union of simple closed geodesics.

Question 11.2.1 (Gromov). *Are nontrivial stationary nets dense on any closed Riemannian surface? In other words, for each non empty open subset U , is there a stationary net intersecting U ?*

The answer is evidently positive for every surface of constant curvature. Indeed, in this case tangent vectors to closed geodesics are dense in the unit tangent bundle and we can take the union of closed intersecting geodesics as a stationary net. On other manifolds, the answer is less trivial, for example because in higher dimensions periodic geodesics may not intersect.

It is also not known whether the union of the images of the closed geodesics is always a dense subset of the manifold. Weinstein showed that any manifold carries a metric (even a bumpy metric) for which the tangent vectors to the closed geodesics are not dense in the unit tangent bundle [Weinstein1970], but his argument says nothing about the projection of the union of the closed geodesics to the manifold. The following question is also still open:

Question 11.2.2. *Are the tangent vectors to closed geodesics dense in the tangent bundle for a generic Riemannian metric on a closed manifold?*

It is known that this is the case for metrics with nonuniformly hyperbolic geodesic flows.

12. HOW TO RECONSTRUCT A METRIC FROM ITS GEODESICS.

A (Riemannian or semi-Riemannian) metric allows one to construct geodesics. Every nonzero vector at every point is tangent to a unique curve from the family of geodesics. This section is dedicated to open problems related to the following questions: given a family of curves with this property, are they the geodesics for a metric; and, if there is such a metric, is it unique and how can it be reconstructed? These questions can also be posed for Finsler metrics, but in that context most natural problems are either solved or seem to be out of reach.

Let us introduce some notation. By a *path structure* we will understand (following [Thomas1925]) a family of curves $\gamma_\alpha(t)$ such that for any point x and for any vector $v \in T_x M$, $v \neq 0$, there exists a unique curve γ from this family such that for a certain t we have $\gamma(t) = x$ and $\dot{\gamma}(t) \in \text{span}(v)$. We think that the curves are smooth and smoothly depend on the parameters $\alpha = (\alpha_1, \dots, \alpha_{2n-2})$. We may insist that the parametrization of the curves stays fixed, or we may be willing to reparametrize them.

The first step is to find an affine connection for which the curves are the geodesics. The problem reduces to a system of linear equations (whether one can actually write down and solve them depends on the form in which the curves are given.) We outline the main ideas.

First consider the case in which the parametrization of the curves is fixed. If all the curves γ from this path structure are affinely parameterised geodesics of a connection $\nabla = (\Gamma_{jk}^i)$, then at any point $p \in M$ the Christoffel symbols satisfy the system of equations

$$(12.1) \quad \ddot{\gamma}(t)^i + \Gamma_{jk}^i \dot{\gamma}(t)^j \dot{\gamma}(t)^k = 0$$

for all curves γ from the path structure such that $\gamma(t) = p$. This is a system of n linear equations in the $\frac{n^2(n+1)}{2}$ unknowns $\Gamma(p)_{jk}^i$. Consequently $\frac{n(n+1)}{2}$ curves γ_α from the path structure give us a system of $\frac{n^2(n+1)}{2}$ linear equations in $\frac{n^2(n+1)}{2}$ unknowns $\Gamma(p)_{jk}^i$. It is an easy exercise to see that if the velocity vectors of the curves at the point p are in general position, then this system is uniquely solvable; by solving it we obtain the Christoffel symbols at this point. Clearly, these Christoffel

symbols should satisfy the equation (12.1) for all curves γ from the path structure (so generic path structures do not come from a connection). It depends on how the curves γ are given whether it is possibly to check this. For example, if all the curves are given by explicit formulas that depend algebraically on t and on α , then this is an algorithmically doable but computationally complicated task.

Note that the above considerations show that reconstruction of a symmetric affine connection from affinely parameterized geodesics is unique.

Let us now deal with the reconstruction of a connection from unparameterized curves. Our goal is to find a connection $\nabla = \left(\Gamma_{jk}^i\right)$ for which each curve γ_α from our path structure, after an appropriate reparameterization, is a geodesic. In this case, a similar idea works. The analog of (12.1) is

$$(12.2) \quad \ddot{\gamma}(t)^i + \Gamma_{jk}^i \dot{\gamma}(t)^j \dot{\gamma}(t)^k = c \dot{\gamma}^i,$$

where the unknowns are $\Gamma(p)_{jk}^i$ and c (though we are interested only in $\Gamma(p)_{jk}^i$). For one curve γ containing $p \in M$ we have therefore n equations in $\frac{n^2(n+1)}{2} + 1$ unknowns $\Gamma(p)_{jk}^i$ and c . For two curves γ_1, γ_2 we obtain then $2n$ equations in $\frac{n^2(n+1)}{2} + 2$ unknowns $\Gamma(p)_{jk}^i, c_1, c_2$ and so on. We see that for $k > \frac{n^2(n+1)}{2(n-1)}$ curves γ_α from the path structure (passing through the point p) we have more equations than unknowns; by solving this system (if it is solvable) we obtain a connection. See [Matveev2012a, §2.1] for more details.

Note that (as was already known to [Levi-Civita1896] and [Weyl1921]) the solution Γ_{jk}^i of (12.2), if it exists, is not unique: two connections $\nabla = \left(\Gamma_{jk}^i\right)$ and $\bar{\nabla} = \left(\bar{\Gamma}_{jk}^i\right)$ have the same unparameterized geodesics, if and only if there exists a $(0, 1)$ -tensorfield ϕ_i such that

$$(12.3) \quad \bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \delta_k^i \phi_j + \delta_j^i \phi_k.$$

Thus, the freedom in reconstructing of a connection is an arbitrary choice of a 1-form ϕ_i .

Connections $\nabla = \left(\Gamma_{jk}^i\right)$ and $\bar{\nabla} = \left(\bar{\Gamma}_{jk}^i\right)$ related by (12.3) are called *projectively equivalent*; projective equivalence of two connections means that they have the same geodesics considered as unparameterized curves.

Let us now touch on the question of whether/how one can reconstruct a metric (Riemannian or of arbitrary signature) from a path structure. As we explained above, it is relatively easy to reconstruct an affine connection (resp. a class of projectively equivalent affine connections) from affinely (resp. arbitrary) parameterized geodesics, so the actual question is how to reconstruct a metric from its affine connection (resp. a class of projectively equivalent affine connections), when it is possible, and what is the freedom. Let us first discuss how/whether it is possible to reconstruct a metric parallel with respect to a given symmetric affine connection.

This question is well-studied; see for example the answers of Bryant and Thurston in [Bry-Thu2011]. A theoretical answer is as follows: the affine connection determines the holonomy group. The existence of a metric with a given affine connection is equivalent to the existence of a nondegenerate bilinear form preserved by the holonomy group, see e.g. [Schmidt1973]. A practical test for the existence of the metric is as follows: the connection allows to construct the curvature tensor R^i_{jkl} , and if the connection is the Levi-Civita connection of a metric, then this metric

satisfies the following equations:

$$(12.4) \quad R^s_{jkl}g_{si} = -R^s_{ikl}g_{sj}, R^s_{jkl}g_{si} = R^s_{lij}g_{sk}.$$

These equations are essentially the algebraic symmetries of the Riemann curvature tensor: the first one corresponds to $R_{ijkm} = -R_{jikm}$, and the second corresponds to $R_{ijkm} = R_{kmi j}$. One should view these equations as linear equations in the unknowns g_{ij} ; the number of equations is bigger than the number of unknowns so it is expected that the system has no nonzero or nondegenerate solution (and in this case there exists no metric compatible with this connection). In many cases the solution is unique (up to multiplication by a conformal coefficient) and in this case we already have the conformal class of the metrics. Now, having the conformal class of the metric we have the conformal class of the volume form and it is easy to reconstruct the metric using the condition that the volume form is parallel, which immediately reduces to the condition that a certain 1-form is closed; see also [Mat-Trau2014].

Note that if the equations (12.4) do not give enough information one could consider their “derivatives”

$$(12.5) \quad R^s_{jkl,m}g_{si} = -R^s_{ikl,m}g_{sj}, R^s_{jkl,m}g_{si} = R^s_{lij,m}g_{sk}$$

which gives us again a huge system of equations in the same unknowns g_{ij} . If necessary one then considers higher order derivatives until there is enough information. The general theory says that in the analytic category the existence of a nondegenerate solution of the resulting system of equations implies the existence of a metric whose Levi-Civita connection is the given one. In the nonanalytic setting, however, there exist C^∞ counterexamples.

Let us now discuss how unique is the reconstruction of a metric from affinely parameterized geodesics. Locally, the answer is known: for Riemannian metrics, if was understood already by Cartan and Eisenhart [Eisenhart1923]; for metrics of arbitrary signature, the answer is in the recent papers [Boubel2012, Boubel2014]. Both results are an explicit local description of all metrics having the same Levi-Civita connection. In the Riemannian case, a global analog of the result of Cartan is due to [DeRham1952]. As the only interesting unsolved problem in this topic we suggest

Question 12.0.3. *Suppose a closed manifold (M, g) admits a nonzero $(1, 1)$ -tensor field that is self-adjoint with respect to g , parallel and nilpotent. Does this manifold or its double cover admit a nonzero light-like parallel vector field?*

Also in the Finsler case parameterised geodesics determine the connection; we will call two Finsler metrics affinely equivalent if any geodesic of the first metric (considered as a curved parametrized such that the length of the velocity vector is a constant) is a geodesic of the second metric.

Problem 12.0.4. *Describe all affinely equivalent Finsler metrics.*

The “unparameterized” versions of these problems for Riemannian metrics are the subject of the recent survey [Matveev2012a]. Roughly speaking, the situation is similar to the one in the “parameterized” case: locally, a general strategy for reconstructing a metric is understood: the existence of a metric compatible with a path structure is equivalent to the existence of (nondegenerate) parallel sections of a certain tensor bundle [Eas-Mat2008]. A connection on this tensor bundle is

constructed by the projective structure constructed by the path structure. Thus, a theoretic answer is to see whether the holonomy group of this connection preserves a certain nondegenerate element of the fiber, and a practical method is to construct the curvatures of the connection and look for elements of the fiber compatible with the curvature. As an interesting open problem we suggest:

Problem 12.0.5. *Construct a system of scalar invariants of a projective structure that vanish if and only if there exists (locally, in a neighborhood of almost every point) a metric compatible with a given projective structure.*

In dimension two, the problem was solved in [Br-Du-Eas2009], and the system of invariants is quite complicated — the simplest invariant has degree 5 in derivatives. It is possible that in higher dimensions the system of invariants could be easier in some ways, since in this case the PDE-system corresponding to the existence of a metric for a projective structure has a higher degree of overdeterminacy. In particular fewer differentiations are needed to construct the first obstruction to the existence of a metric class. See the recent paper [Dun-Eas2014].

Let us now discuss the freedom in reconstructing the metric by its geodesics; an equivalent question is how many different metrics can have the same geodesics considered as unparameterized curves. Of course, the metrics g and $\text{const} \cdot g$ have the same geodesics; in [Matveev2012a] it was shown that for a generic metric any projectively equivalent metric is proportional to it with a constant coefficient. There exist local (the first examples are due already to Lagrange, Beltrami and Dini) and global examples of nonproportional projectively equivalent metrics. Locally, in the Riemannian case, a complete description of projectively equivalent metrics is due to Levi-Civita [Levi-Civita1896] and in arbitrary signature is due to [Bol-Mat2013]. Globally, in the Riemannian case, the situation is also pretty clear, but if the metrics have arbitrary signature, virtually nothing is known. A general problem is to understand topology of closed manifolds admitting nonproportional projectively equivalent metrics of indefinite signature and as the simplest version of this problem we suggest

Question 12.0.6. *Can a 3-dimensional sphere admit two nonproportional projectively equivalent metrics of indefinite signature?*

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