Geometry and topology of Artin groups

organized by
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Workshop Summary

Goals of the Workshop

The workshop focused on Artin groups, an important and quite mysterious class of groups which have been studied from many different perspectives (algebraic, combinatorial, geometric, etc). They have rich connections to mapping class groups, hyperplane arrangements, Coxeter groups, and geometric group theory.

There have been some recent breakthroughs in this area, and the goal of this workshop was to bring together experts who were studying Artin groups from different perspectives in order to understand these breakthroughs and plot new directions of research.

There were two talks each morning. The topics on Monday and Tuesday were more focused on background (though certainly contained some recent developments) whereas the other talks were specifically focused on current research. On Monday afternoon, there was a problem session moderated by Jing Tao. There were 46(!) research problems included in this conference, further evidence of the flurry of activity in this area.

Working groups

Tuesday-Friday afternoons consisted of working in small groups of 3-7 people. Some groups lasted the entire week while others sprang up mid-week, and there was a good deal of mixing between the groups as the week went on. We will give a short summary of the research that each group worked on, potential progress that the groups made, as well as any future plans.

Dual Artin groups.

A *dual Artin group* has a presentation determined by a Coxeter group W and a choice of a Coxeter element $w \in W$. Conjecturally, each dual Artin group is isomorphic to the associated Artin group A_W. This group started by going over the process by which one finds a dual presentation of an Artin group and discussing how this procedure could be applied to one of the simplest unknown examples: the right-angled Artin group with a pentagonal defining diagram. The group directly explored the structure of factorizations of the Coxeter element in the hyperbolic plane tiled by right-angled pentagons, convincing themselves that various categories of reflections all occur and that there are also lots of reflections that occur in reflection factorizations which are far from the glide axis of the Coxeter element. By the end of the week we started shifting our attention to the technical algebraic tools (Wall forms and root games) that would facilitate a more serious attempt to solve the problem of constructing a dual presentation and proving that it defines a group which is indeed an Artin group. There was a core group of people that attended every afternoon and there was general agreement that we would continue pursuing this question after the conclusion of the conference.

Helly groups.

A graph is *Helly* if every family of pairwise intersecting combinatorial balls has a nonempty intersection, and a group is Helly if it acts geometrically on a Helly graph. This has very strong consequences for G (bi-automaticity, existence of EZ-boundary, etc) and whether Artin groups are Helly is actively studied. In the problem session, Thomas Haettel asked specifically whether the Artin group $\widetilde{A}_2$ was Helly. Here, the corresponding Coxeter group is the group generated by reflections in an equilateral Euclidean triangle, and Hoda showed this latter group was *not* Helly.

The group discussed mostly the nonpositively curved geometry of the $\widetilde{A}_2$ affine Artin group A. They reviewed why A is a CAT(0) group, why $A \times \mathbb{Z}$ is Helly, and why A is not compactly cubulated. They discussed a convincing strategy to show that A is not Helly, and also started to investigate whether A could or not act properly (and not cocompactly!) on a CAT(0) cube complex.

Intersection of parabolics.

For a general Artin group A with generating set S, the subgroup generated by a subset $T \subset S$, called a *special parabolic subgroup*, is isomorphic to the Artin group A_T. The conjugates of the subgroups A_T are called *parabolic subgroups.* In the problem session, it was asked whether intersections of parabolic subgroups are themselves parabolic. This is known for Coxeter groups, and for some special classes of Artin groups, e.g. spherical type Artin groups.

The group considered the special case where the Artin group A splits as an amalgamated product of two special parabolic subgroups with this intersection property. If such amalgamated always had the intersection property, it could inductively be applies to the Artin groups of FC-type, which corresponds to the defining nerve being a flag complex. It was previously known by work of Morris-Wright and Paris that this intersection property in FC-type was true if one of the parabolic subgroups was spherical, but the general case remained unknown. The strategy attempted by the group utilized the action of the group on the associated Bass-Serre trees and analysis of the subspaces fixed by the parabolic subgroups. The group outlined the proof for pure parabolics, i.e. the analogously defined subgroups of the pure Artin group, but the general case requires some new ideas.

Automorphisms of Artin groups.

Automorphism and outer automorphism groups of right-angled Artin groups are well-understood, but little is known about automorphism groups of other Artin groups. A starting point for this investigation would be to find a set of generators for these automorphism groups (analogous to the Laurence-Servatius generators for RAAGs). The group considered this question for Artin groups of even type (that is, all edges have even labels). They came up with a conjectural generating set for these automorphism groups and plan to pursue this question further.

Biautomaticity of Artin groups.

The aim of the group was to prove 2-dimensional Artin group are biautomatic. During the discussion, two of the group members, Damian Osajda and Piotr Prytycki, explained their previous work proving the biautomaticity of 2-dimensional Coxeter groups. The group then went through a construction of adjusting the normal form in the Coxeter group case to 2-dimensional Artin groups. While this normal form is natural, it had a problem of failing the fellow-travel property in simple examples. There was some discussion on what should be the most natural construction of a modified normal form, but in the end, the normal form became more and more complicated, and there seemed to be too many cases to analyze. At the end of the discussion, there were several other suggestions of what could be an alternative normal form, for example, considering path systems in the Deligne complexes, and then trying to lift them to the Artin groups. However, each of the proposed normal forms also had trouble with the fellow travel property. There was also a related discussion, including how to generalized normal forms to higher dimensional Artin groups. In the end, the group concluded that the problem of biautomaticitiy of 2-dimensional Artin group is probably hard, and it is not clear what should be the ideal normal form to consider. The group does not plan on any further meetings, though made some progress in the sense that, through the discussion, it became clear what are the problems with several existing normal forms on 2-dimensional Artin groups.

The isomorphism problem for Artin groups.

The general question that this group attacked is to determine precisely when two different presentation graphs yield two isomorphic Artin groups. The group focused on the special case of two-dimensional Artin groups (one of the group members, Nicolas Vaskou, had recently proved more specific large-type case). In the first meeting, the group reviewed the proof of Vaskou, identifying in particular the various points where the hypothesis of being large-type is used. The group then worked on generalizing that proof to a few small examples of more general two-dimensional Artin groups, which required working on a larger class of parabolic subgroups than that originally considered by Vaskou in the large-type case. The group later focused on the more rigid (and more manageable) case of even-type two-dimensional Artin groups, where we obtained a rough sketch of proof that these groups are uniquely determined by their presentation graphs. The group plans to continue working on this problem in the general two-dimensional case, working out the details in the even-type case, and generalizing the arguments to allow for odd labels.

Deligne complex of A_{B_4}.

Conjectually, the Deligne complex associated to any Artin group has a natural CAT(0) metric, analogous to the CAT(0) metric on the Davis complex constructed by Moussong. The goal of this group was to show that the "spherical Deligne complex" of the Artin group A_{B_4} endowed with the Moussong metric is CAT(1), which immediately implies the conjecture for this case. To do this, the group needs to rule out the existence of closed geodesics of length $< 2\pi$. Their progress so far has been to show that all but one of the edge paths of length $< 2\pi$ fail to be geodesic. What remains is to show that this remaining edge loop is not a geodesic, and to show that non-edge paths of length $< 2\pi$ are not geodesic. The group has a rough argument for the last part, and just needs the details verified. The group feels like they are quite close to a solution, and plan on continuing work on this problem in late

October.

Outcome

The working groups addressed some new and some ongoing problems. While no major break- throughs occurred during the week of the workshop, many new ideas were discussed and participants learned about a wide variety of techniques which may potentially help to solve these problems. Quite a few new collaborations were formed and we expect these discussions to be ongoing in the coming months.