Geometric partial differential equations from unified string theories

organized by
Tristan Collins, Anna Fino, and Duong H Phong

Workshop Summary

Introduction.

This workshop focused on new directions in geometric partial differential equations arising from unified string theories. Geometric partial differential equations arising from unified string theories are expected to have deep connections to underlying geometric structures. These structures are expected to form finite dimensional moduli spaces, and may lead to the discovery of exciting new geometric invariants.

From a purely analytic view-point, PDEs arising from unified string theories exhibit many new features, including the presence or sources and fluxes, nonlinear "free boundaries", and connections to notions form generalized geometry. Beyond making progress on key research questions, a primary goal of the workshop was to identify a list of properly formulated mathematical problems and directions for research which might guide progress in the area over the years to come.

Summary.

The main topics discussed in the workshop were as follows.

- The type IIA and type IIB vacuum equations in the presence of sources; local and global solutions; incompleteness and free-boundaries.
- Geometric flows of the heterotic string and $SU(3)$ structures; connections to generalized geometry and string algebroids; infinite dimensional moment maps in generalized geometry and connections with algebraic notions of stability.
- Aspects of non-Kähler geometry, and geometry of non-Kähler Calabi-Yau manifolds; stability of the $\partial\bar{\partial}$-lemma under conifold transitions; Reid's fantasy and connections with the heterotic string.

Each day, as per the usual AIM format, the workshop consisted of two talks followed by problem sessions, as well as allowing for time for informal discussion.

Talks.

The talks served both to provide background and surveys, as well as give more specific talks on results, including showcasing of very recent work by junior participants.

Alessandro Tomasiello gave a survey of geometric PDEs from unified string theories from a physics point of view.

Li-Sheng Tseng gave a survey of geometric PDEs from unified string theories from the mathematical point of view.

Mario Garcia-Fernandez gave a talk on Higher instantons and the Hull-Strominger system and connections with Reid's fantasy.

Anthony Ashmore gave a talk on the Hull-Strominger system, involutivity and moment maps.

Simon Brendle gave a talk on the Horowitz-Myers conjecture and a new positivity energy theorem.

Luigi Vezzoni spoke about geometric flows of closed forms, including flows for the type IIA and IIB strings, flows in G_2 geometry, and new techniques for proving short time existence.

Vicente Cortés spoke about the Darboux theorem for generalized complex structures on transitive Courant algebroids.

Carlos Shahbazi gave a talk on the three-dimensional parallel spinor flow, which is the evolution flow defined by a parallel spinor on a globally hyperbolic Lorentzian four-manifold. Carlos also spoke about extensions of these ideas to super-gravity theories.

Adriano Tomassini spoke about cohomological properties of non-Kähler manifolds, including relations between Bott-Chern and de Rham cohomology groups.

Kuan-Hui Lee spoke about results from his forthcoming thesis on the dynamical stability of generalized Ricci solitons and non-Kähler Calabi-Yau metrics along generalized Ricci flow.

Problem sessions.
The problem sessions were very productive, with participants exploring the different topics, having engaging discussions, and forging new academic relationships. The main problems discussed were as follows.

(1) **Can one construct solutions of the type IIA equations with singular sources?** The group discussed the possibility of solving the vacuum equations of motion for the type IIA string in $\mathbb{R}^6$, with its standard symplectic structure, but with sources on intersecting Lagrangian planes. The existence of such solutions are expected by physicists, but have not been constructed even in the physics literature. The approach was to look for highly symmetry configurations and then reduce the problem to a system of coupled PDEs. While significant progress was made, it became clear that the resulting system of PDEs was unlikely to substantially simplify, and computer algebra systems would be needed to make meaningful progress. People involved in the project were T. Collins, F. Tong, M. Garcia-Fernandez, A. Fino, A. Tomasiello, A. Tomassini, L. Vezzoni, and A. Ashmore.

(2) **Can one construct solutions of the type IIA equations with sources on** T^*S^3**?** The group discussed the possibility of solving the type IIA equations on T^*S^3 with a single source located on the zero section of the bundle. By exploiting the symmetry of the underlying manifold, a manageable ODE system was obtained. T. Collins, M. Garcia-Fernandez, F. Tong, A. Tomasiello, and A. Ashmore and continuing to pursue this problem, with the hope of using these model solutions as pieces in gluing construction of compact solutions of the type IIA equations with sources.

(3) **Is the $\partial\bar{\partial}$-lemma stable under conifold transitions?** The discussion of the validity of $\partial\bar{\partial}$-Lemma under conifold transitions evolved into discussion of possible complex manifolds in which the Lemma could be checked. Among the examples are the twistor spaces of Riemannian self-dual 4-manifolds. It is known that among them only two are Kähler, and some are Moishezon. A quick check leads to the

conclusion that $\partial\bar{\partial}$-Lemma on a twistor space could be satisfied only if the base manifold is simply connected and with positive or vanishing intersection form, which leaves the 4-sphere and connected sums of complex projective planes. On all of these 4-manifolds there are self-dual metrics for which the twistor space is Moishezon and hence satisfies the $\partial\bar{\partial}$-Lemma. Since the property of a compact complex manifold to satisfy $\partial\bar{\partial}$-Lemma is open, there are twistor spaces which satisfy the Lemma, but which are not Moishezon. Because the $\partial\bar{\partial}$-Lemma is related to the Bott-Chern and Aeppli cohomology, a natural project is to calculate them for some examples, and the twistor space of a flat 4-torus is a natural candidate. G. Grantcharov, A. Tomassini, and N. Tardini continued working on this question during the workshop and expect this project to result in a new publication. Other people involved in the discussion include A. Fino, L. Vezzoni, F. Tong, and L-S. Tseng.

(4) **Generalized $G_2 \times G_2$ and $SU(3) \times SU(3)$ and Hitchin-type flows** This group discussed generalizations of the Hitchin flow that occur in supergravity theories. The group computes the supersymmetry equations for a 10-dimensional space time arising rom a product of an internal 7-manifold and Minkowski 3 space in type II. The resulting equations describe a $G_2 \times G_2$ structure on the generalized tangent bundle $TM \oplus T^*M$. This structure can be decomposed in terms of an $SU(3) \times SU(3)$ structure, and this can be interpreted as a flow. The group noted that in work of Passias and Prins for a standard G_2 structure, where one sees that the existence of a solution is reduced to a condition on the intrinsic torsion. This condition also appears in heterotic solutions, and the group conjectured that this arises from S-duality to a solution with only NSNS three-form flux. The group focused on the examples considered by Passias and Prins, and in particular a case on the product of S^3 with a hyperKähler manifold. Participants in this group included A. Tomasiello, A. Fino, A. Ashmore, and C. Shabazi.

In summary, important progress was made towards each of these problems and it seems very likely that these discussions will yield concrete progress.

Conclusion.

The organizers were extremely pleased with the results of the workshop, which greatly exceeded their expectations. As well as the participants learning a lot about the area, there was tangible progress on numerous fronts, new collaborations were developed, and the workshop is certain to lead to significant results for the field in the near future.

The organizers are grateful to the AIM staff for providing an extraordinary level of support for this very successful workshop.