Geometric realizations of Jacquet-Langlands correspondences

organized by
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Workshop Summary

Overview

This workshop was devoted to topics related to geometric constructions realizing instances of Langlands functoriality. The main focus was on the preprints of Ichino-Prasanna and Xiao-Zhu and closely related topics.

The workshop followed closely the proposed AIM format. Talks were given in the morning. One of which was Yifeng Liu's talk which described a new case of geometric Jacquet-Langlands exploiting Picard modular surfaces at primes of bad reduction associated to the unitary group in three variables, as extracted from the upcoming work of Liu-Tian-Xiao-Zhang-Zhu.

More generally, the bulk of the talks were focused on the central themes of the workshop: Langlands functoriality, Hodge and Tate conjectures, as well as examples from the preprints mentioned above. Some ideas and conjectures discussed have not yet been published (some are available in written form as preprints such as Lafforgue-Zhu on the Kottwitz conjecture, and also works already mentioned; but not all, such as Drinfeld's mysterious (complex of) coherent sheaves in relation with the cohomology of Shimura varieties), so participants were exposed to the latest formulation of some recent contributions to the area.

A long-standing thorny issue in the area of algebraic cycles has been the dearth of techniques to construct them beyond the classical, key examples related to Frobenius (over finite fields), abelian varieties and of course Shimura varieties. Two talks were given on general techniques closely related to algebraic cycles: one of which by Anna Cadoret who explained Yves André's theory of "cycles motivés" and another by Héléne Esnault who mentioned part of a (yet incomplete) theory of p-adic deformation theory of algebraic cycles with an application to K3 surfaces.

Afternoons were spent working on projects, whose number varied between 5 and 7. We summarize below those that lasted at least 3 out of the 4 afternoons ascribed to such activities.

Working Groups

(1) *Comparison between characteristic zero à la Ichino-Prasanna and characteristic p constructions of Xiao-Zhu*

This team studied the relation between cycles in characteristic zero and characteristic p. Recall that the work of Xiao-Zhu (XZ) constructs cycles in characteristic p that exhaust all the expected Tate cycles under certain genericity assumptions. On the other hand, the work of Ichino-Prasanna (IP) constructs Hodge-Tate cycles in characteristic zero. These Tate cycles can be reduced mod p and it is natural to ask how they relate to the XZ cycles. It turns out that in all situations where the work of IP applies, the space of Tate cycles in characteristic p is larger than the space of Tate cycles spanned by the algebraic cycles constructed in XZ. So it is extremely natural to ask if the reduction of the IP cycles is contained in the span of the XZ cycles, or if they lie outside this span. Interestingly, the team was able to check in some situations that the reduction actually lies outside the span. For example, one case considered was that of a real quadratic field F, a prime p inert in it, a Shimura variety $X_1 \times X_2$ where X_1 and X_2 correspond to quaternion algebras over F split at both infinite places, and an automorphic representation $\pi_1 \boxtimes \pi_2$, where π_1 and π_2 are automorphic representations of the two quaternion algebras that are Jacquet-Langlands transfers of each other. In this case, the span of the Tate cycles in characteristic p is six dimensional, while the XZ space is four dimensional. The team checked (using a Galois theoretic argument) that the the reduction of the IP cycle (along with a partial Frobenius translate of it) generate a complementary two dimensional space.

The team discussed various new directions that might be explored such as computing the intersection pairing of the IP cycles with the XZ cycle. There was some hope that the intersection numbers might be related to period integrals of automorphic forms.

(2) *Geometric realizations of other instances of functoriality than Jacquet-Langlands*

The preprints of Ichino-Prasanna and Xiao-Zhu dealt with the most basic case of functoriality, namely, the Jacquet-Langlands transfer from one group to its inner form, but there are many other cases of functoriality whose existence has been proved automorphically. This team studied the case of endoscopic transfer from $\mathrm{U}(1,1)$ to $\mathrm{U}(2,1)$. In characteristic zero, the theory of Kudla-Millson gives us a lifting of a cusp form on $\mathrm{U}(1,1)$ of weight 3 to a codimension 1 cycle on a $\mathrm{U}(2,1)$ Shimura variety. However, its connection to the formulation by Ichino-Prasanna needs to be clarified. The team also tried to find ambient Shimura varieties which extend the work of Tian-Xiao (for quaternionic Shimura varieties) and Xiao-Zhu in characteristic $p > 0$ to the endoscopic case by using the usual fibration trick over a finite basis.

(3) *Around the plectic conjecture of Nekovář and Scholl*

The plectic conjecture of Nekovář and Scholl predicts that when the Shimura variety is attached to a reductive group obtained by Weil restriction of scalars from a totally real field F to $\mathbb{Q}$, its cohomology should admit additional Galois symmetry (the action of the plectic Galois group) extending the usual Galois symmetry. In addition, this additional symmetry should be "motivic" in suitable sense. This team examined the corresponding conjecture in the function field case, where Shimura varieties are replaced by moduli of Shtukas. In addition, the team realized that when considering the cohomology of mod p fibers of such Shimura varieties, the shadow of the additional symmetry has already been (partially) encoded in Xiao-Zhu's work and Zhu's unpublished work on the $S = T$ conjecture. Namely, Xiao-Zhu constructed an action of certain algebra on the cohomology of the mod p fiber of a Shimura variety (with good reduction at p) by correspondences. By the $S = T$ conjecture, the action of this algebra contains the usual action of the spherical Hecke algebra at p. But it also contains the action of partial Frobenii. In addition, if the prime p is totally inert in the totally real field F, the algebra also contains the symmetric group S_d where $d = [F : \mathbb{Q}]$. All of these facts are consistent with the plectic conjecture.

(4) *Geometry of Affine Deligne-Lusztig Varieties: the* $\mathrm{U}(3,2)$ *case*

This team studied irreducible components of affine Deligne-Lusztig varieties related to Shimura varieties for the unitary group U(2,3). According to a result of Xiao-Zhu, there are two types of irreducible components in those affine Deligne-Lusztig varieties. The first involves only minuscule characters, and the other involves a non-minuscule character. The team showed that the first variety is isomorphic to the Fermat 3-fold. For the second one, the team constructed a morphism from an open subscheme of the irreducible component to a product of two Grassmannian varieties. Further the team showed that the morphism factors through a 3-dimensional subspace which is birational to the Fermat 3-fold.