The geometry of polynomials in combinatorics and sampling

organized by
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Workshop Summary

Summary

Workshop activities consisted of nine talks on topics including the general theory of the geometry of polynomials and applications in graph theory, combinatorics, sampling, and probability. There were several active working groups that remained fairly stable throughout the week. Individual reports from these groups are below.

Reports from Working Groups

Spectral Independence of matchings.
(written by Kuikui Liu)

Our group originally worked on disproving Kahn's conjecture negative correlation for the uniformly random forest in graphs. We switched to studying spectral independence for matchings on Thursday. That day, Kuikui stated the motivations and definitions more precisely, and gave an overview of the previous techniques that were used in the bounded-degree case. We also discussed possible approaches based on decomposing the distribution by conditioning on whether or not a given selection of vertices are matched, with the hope of using the fact that the sum of vertex-to-vertex influences remains bounded by 1, independent of the degrees. On Friday, we discussed the classical canonical paths approach due to Jerrum-Sinclair, which is more combinatorial, and yields a polynomial-time sampler even if the degrees are unbounded. We played around with modifications to their scheme based on a random choice of paths, similar to how one can recover the optimal spectral gap for the uniform distribution over the hypercube by performing bit flips with respect to a random ordering of the coordinates.

Recently, spectral independence for matchings was solved by researchers not present at this AIM workshop (see arXiv:2504.03406). However, we intend to continue studying the question of whether or not there exists a nearly-linear time algorithm for sampling matchings, which was the original algorithmic motivation.

Overlapping Sets of matroids.
(written by Daniel Qin)

Lorentzian polynomials are a powerful tool to obtain surprising log-concavity in algebraic combinatorics. We study the following generating polynomial of a combinatorial statistic given by a matroid and sequence of subsets. Fix a matroid $M = ([m], \mathcal{B})$ of rank d and a sequence of subsets $S = (S_1, \dots, S_n)$ where $S_i \subseteq [m]$. For each $\alpha \in^n$, $c_\alpha(S)$ is the number of bases $B \in \mathcal{B}$ such that there exists a sequence of disjoint subsets $T_i \subseteq S_i$ that cover B where $|T_i| = \alpha_i$. This allows us to define the polynomial:

$$P_M(S;x) = \sum_{\alpha \in ^n} c_\alpha(S) \frac{x^\alpha}{\alpha!}.$$

[Proposed by Chris Eur] Prove that for any M and S, $P_M(S;x)$ is Lorentzian. The following results are some that were already known by Eur:

(1) The statement holds for any direct sums of rank 1 matroids M and any S.
(2) The support is $M-$convex.

We have the following results after the week:

(1) We (re)proved that the support is $M-$convex.
(2) We proved that statement holds for any matroid M and disjoint sequence S.
(3) We produced code to generate examples of $P_M(S;x)$ for small matroids and small sets.

In addition to the above results, we understand how the generating polynomial of interest behaves under some natural operations on sequences of sets, e.g., duplicating singletons. We also have promising ideas for how to prove the statement for $U_{2,m}$ and any S, which we hope extend onto $U_{d,m}$.

Zeros of chromatic polynomials.
(written by Guus Regts)

We had one short (approx 1.5 hr) meeting in which we discussed a variation on the approach given in the lecture by myself (Guus Regts) on Friday morning. Basically the idea was to try to start at a graph with known chromatic polynomial (such as a chordal graph) and reduce to smaller graphs inductively trying to preserve properties of its zeros. While the idea seems nice, it was not clear to us how to pursue this, partly because we would have to overcome certain subtleties related to the fact that zeros should grow in modulus since complete bipartite graphs could be obtained in this recursion. At the end of the meeting we also discussed how to prove zero-freeness of other graph polynomials connected to proper colorings.

Even jump systems.
(written by Cynthia Vinzant)

This group generated and worked through several potential ideas for generalizing the theory of Lorentzian polynomials to even jump systems, which are the the "type-B" analogue to M-convex sets and include combinatorial objects such as Δ-matroids. Part of this task involved formulating a "wish list" of desired properties for such a class of polynomials, which included preservation under multiplication and symmetrization. These would in turn give a useful theory of preserving operations that is crucial in the theory of stable and Lorentzian polynomials. Another part of the "wish list" is that the generating polynomials of Δ-matroids (or even jump systems more generally) are 1/2-log-concave. This is a notion developed by Anari et. al. on fractional log-concavity that has strong implications for the efficacy of sampling algorithms. One potential implication of this theory would be the log-concavity of some discrete sequences associated to Δ-matroids, which is currently not known or even conjectured to be true. While there were counterexamples to many of the proposed classes of polynomials, several interesting open questions remain.

Roots of Chow polynomials.
(written by Jacob P. Matherne)

A number of polynomials associated to matroids are conjectured to be real-rooted: they are the Kazhdan–Lusztig polynomials, the Z-polynomials, the Chow polynomials, and the augmented Chow polynomials. This group focused on the problem of real-rootedness of Chow polynomials.

A sizeable portion of the week was spent trying to prove that Chow polynomials of braid matroids are real-rooted. This is an important infinite family of matroids, and these Chow polynomials have a very explicit formula in terms of descents, due to Stump. Brändén and Vecchi recently proved that Chow polynomials of uniform matroids are real-rooted; our group tried to emulate what they did, but now for braid matroids, by using Stump's explicit formula. Although we were able to chase the problem down to a few lemmas, we were unable to verify them, so ultimately this approach was unsuccessful.

Our group also spent some time investigating stability and Lorentzianity of various multivariate analogues of Chow polynomials for arbitrary matroids. The normalized polynomial whose terms are the Feichtner–Yuzvinsky basis elements of the Chow ring does not have M-convex support. Also, the multivariate Chow polynomial of Hoster is not Lorentzian nor denormalized Lorentzian. (This fails for some small uniform matroids.) It would be good to look for other natural multivariate analogues of Chow polynomials that may be (denormalized) Lorentzian and/or stable.

Finally, we briefly discussed the problem of real-rootedness for Chow polynomials of arbitrary matroids. More specifically, we considered the Chow polynomials of the family of graphic matroids associated to unicyclic graphs. There were attempts at proving real-rootedness using some deletion formulas involving Chow polynomials, but this endeavor was unsuccessful.

Erhart Volume Conjecture.
(written by Aryaman Jal)

Ehrhart made the following conjecture in 1964: if K is a convex body in ^{n} with the origin as its barycenter and no other interior integer point, then $\text{vol}(K) \leq (n+1)^n/n!$ The relevance of this problem to the geometry of polynomials is twofold. First, the RHS of the inequality can be written as $1/\text{vdW}(n+1)$ where $\text{vdW}(n)$ is the function that appears as the lower bound of the van der Waerden conjecture on the permanent of a doubly stochastic matrix. Although settled by Egorychev, and independently, Falikman in 1981, a new (and drastically simpler) proof of this inequality was derived by Gurvits who appealed to the geometry of polynomials and the notion of the capacity of a multivariate polynomial. The second parallel between these conjectures is the equality case: on the convex body side, equality holds iff the convex body is the unimodular image of the standard simplex scaled by $n+1$; on the permanent side, equality holds iff the matrix is $1/n$ times the all ones matrix.

Our discussion focused on deepening this parallel: we paired Grünbaum's lemma (in convex geometry) with an elementary result in the geometry of polynomials that bounds the capacity of a univariate, real-rooted polynomial in terms of its derivative. The reason for doing so is that both inequalities involve the faction $(n+1/n)^{n+1}$. We couldn't immediately go from the latter result to the former, but we were encouraged by the fact that a well-known approach to the Grünbaum lemma involves the log-concavity of a particular volume function. We then considered Gurvits' "product" polynomial and recognized it as the volume polynomial of boxes. We agreed that formulating an accurate "mixed volume" generalization of the Ehrhart conjecture would be a useful stepping stone to involving further methods from the geometry of polynomials.

Posynomials.
(written by Nima Anari)

We discussed a conjectured generalization of a result of Gårding. Suppose we are given a posynomial, which is a finite sum of the form $\sum c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ where α can range over vectors in ${}^n_{\geq 0}$ and $c_\alpha \geq 0$. This sum can be interpreted as a function f on $(\mathbb{C} - \mathbb{R}_{<0})^n$ by taking a branch of the power functions compatible with the usual ones on ${}_{\geq 0}$. The conjecture is that if our posynomial f is homogeneous (all α have the same $\|\alpha\|_1$), and does not vanish whenever $\mathrm{Re}(z_i) > 0$ for all i, then $\log f$ must be concave over ${}^n_{>0}$. The result of Gårding establishes this when the powers are integral, that is f is an actual polynomial. We discussed an unwritten proof of this conjecture for the case of $n = 2$, and tried to generalize the approach to higher values of n. Homogeneous bivariate polynomials can be factorized into a product of linear terms, and we showed that for $n = 2$, there is a "shadow of a factorization" that can be constructed via potential theory. Roughly speaking we could show that $\log |f(z_1, z_2)|$ can be written as $\int \log |\lambda_1 z_1 + \lambda_2 z_2| d\mu(\lambda)$ where μ is some measure on ${}^2_{\geq 0}$.

We explored various different proofs of Gårding's result to see if they can be generalized to posynomials. The main issue we encountered was that these proofs boil down to a one or two dimensional restriction of f (because one can check concavity by checking it along a line). When restricting a posynomial to a one/two-dimensional space, we get a posynomial if the space is spanned by orthogonal vectors from ${}^n_{\geq 0}$, but not otherwise. In the former case, we could prove the desired concavity, but as soon as we lost the posynomiality, lots of needed ingredients seemed to turn out either false or at least the proofs we had in mind broke down. So in the end, we could establish concavity along some special directions (by essentially reducing to the $n = 2$ case), but the general conjecture remains open.

Negative Correlation Conjecture for Forests.
(written by Marcus Michelen)

Given a finite connected graph G, let F be a forest chosen from G uniformly at random. The negative correlation conjecture asks whether it is the case that for any two edges e and f, we have that the probability that both e and f are in F is at most the product of the probability that e is in F and f is in F. The analogous statement where F is replaced by a random spanning tree is true, and a modern-day proof of it may be given via the determinantal structure of random spanning trees. For forests, much of the discussion at AIM centered around hunting for counterexamples. In particular, we discussed examples on which there is counterintuitive behavior and brainstormed ideas for how one may be able to use such small examples as gadgets to construct a counterexample to the negative correlation conjecture. No counterexample was constructed. We also discussed a heuristic for why the conjecture appears to potentially be true for dense random graphs.

Preservers of real rooted univariate polynomials with trace zero.
(written by Kevin Shu)

We gave a conjecture regarding (diagonal) linear maps from the vector space of polynomials of degree at most d where the coefficient of $t^(d-1)$ is 0 to itself where the image of every real rooted polynomial in the domain under this map is real rooted. This conjecture states that a diagonal linear map has this property if and only if the image of the polynomial $(t-1)^{(d-1)}(t+d-1)$ is real rooted, with $d-1$ roots having the same sign. We went over the proof of the classic Schur-Polya theorem via the Grace-Szego-Walsh theorem and tried to draw analogues between this conjecture and that proof. We gave some analogues of the Grace-Szego-Walsh theorem which would imply this conjecture if they held, but were unable to give an analogous proof which would hold in this setting.