Gibbsian line ensembles

organized by
Evgeni Dimitrov, Milind Hegde, Xuan Wu, and Zhengye Zhou

Workshop Summary

Overview

The study of the Kardar–Parisi–Zhang (KPZ) universality class has advanced tremendously over the past 10–15 years. Central to this progress has been the study of Gibbsian line ensembles: collections of interacting random curves that satisfy a local resampling rule, or domain Markov property. Gibbsian line ensembles have also emerged beyond the KPZ universality class, including as putative scaling limits of level lines in certain random surface models, such as the 3D Ising model and a related proxy, the Solid-On-Solid (SOS) model, under appropriate boundary conditions. The workshop brought together researchers from different perspectives and areas to explore Gibbsian line ensembles and to foster connections among the various directions in which these objects are being studied.

Morning Lectures

Below is a list of the speakers who delivered the morning lectures and a brief description of the talk, in the order that the talks were given. So the first two were the first and second talks on Monday, the next two were the same on Tuesday, and so on.

- **Alan Hammond**: This lecture introduced the resampling rule for Gibbsian line ensembles and several random systems. It traced the developments leading to the Airy line ensemble and discussed its main properties.
- **Shirshendu Ganguly**: This lecture introduced area-tilted line ensembles through their origin as putative scaling limits of level lines of various statistical mechanical models such as the Ising model in 2 and 3 dimensions (with specific boundary conditions) and the Solid-on-Solid model in 2+1 dimensions. It then discussed various properties and proof techniques of the infinite-volume area-tilted line ensembles, including uniqueness and quantitative mixing and correlation decay estimates which were achieved by a certain "reverse coupling" idea.
- **Evgeni Dimitrov**: The lecture introduced last-passage percolation and the corner-growth model as basic examples in the KPZ universality class. It explained how the Robinson–Schensted–Knuth correspondence reveals a discrete line ensemble with determinantal and Gibbs structures, which can be used to prove convergence to the Airy line ensemble. The lecture also discussed extensions to half-space models, where Pfaffian point processes and boundary interactions arise.
- **Li-Cheng Tsai**: This lecture presented a process-level large deviation principle for the stochastic heat and KPZ equations. It described the associated variational problem, the optimal fluctuation governing rare events, and connections with integrable PDE methods.

- **Alexander Dunlap**: This lecture introduced a multilayer extension of the stochastic heat and KPZ equations starting from continuum directed random polymers. Determinantal formulas and the Karlin–McGregor structure were used to describe the resulting interacting line ensemble.
- **Theo Assiotis**: This lecture constructed the limiting line ensemble obtained from the edge-scaled singular values of Brownian motion on $\mathrm{GL}_N(\mathbb{C})$. The main result characterizes its dynamics through an infinite system of SDEs with logarithmic interaction and establishes the Brownian Gibbs property.
- **Lingfu Zhang**: This lecture introduced the Airy beta line ensemble and described its construction as the edge-scaling limit of matrix Brownian motion. It also discussed a characterization of the Airy beta line ensemble using its Stieltjes transform.
- **Evan Sorensen**: This lecture discussed invariant measures for half-space geometric last-passage percolation. The main result classified the extremal invariant measures through stationary models, using a one-force–one-solution principle to describe how the system selects its long-term behavior.
- **Yujin Kim**: This lecture examined a low-temperature solid-on-solid interface constrained by a hard wall. It focused on the geometry and fluctuations of the interface and on the emergence of the characteristic KPZ scaling under appropriate spatial and temporal rescaling.
- **Zoe Himwich**: This lecture explained how the directed landscape can be viewed as a black noise in the sense of Tsirelson. It connected the landscape's local randomness and composition structure with the behavior of geodesics, showing that its randomness cannot be generated from a classical Gaussian component.

List of open problems

1. Construct a Pearcey line ensemble.
2. Study lower-tail large deviations for general line ensembles. For example, under a single-point lower-tail condition, what is the resulting shape of the line ensemble?
3. Consider a two-dimensional self-avoiding random walk tilted by an exponential of its winding number. Analyze the resulting line ensemble on the boundary.
4. Classify Gibbsian line ensembles on $\mathbb{R}$. In particular, can global profiles such as $-|x|^{3/2}$ and $-|x|^4$ occur?
5. Are all extremal line ensembles determinantal? How can one characterize the kernels of line ensembles satisfying the Brownian Gibbs property?
6. Determine the correlation decay of the kth curve of the Airy line ensemble and other related line ensembles.
7. Construct line ensembles for heavy-tailed models. Is the local behavior of the top curve described by fractional Brownian motion? What are the scaling exponents, and what should the local limit be?
8. Study last-passage percolation on the Gaussian Free Field. Can one construct an associated line ensemble and describe the fluctuations of its curves? Is there a connection to Liouville quantum gravity?
9. Can non-integrable perturbations of integrable models provide a tractable source of non-integrable models?
10. What are the dynamics of the Airy line ensemble?
11. Gibbs property in Toda conformal field theory?

Developments on the open problems which have been discussed by the participants

- **Pearcey line ensemble.** The group studied the construction of the Pearcey line ensemble, a bi-infinite family of continuous curves whose point configuration, when restricted to any finite collection of time slices, forms a determinantal point process governed by the extended Pearcey kernel. The group identified a suitable family of models whose scaling limit is expected to produce this ensemble and made progress on several key technical challenges, including tightness and the issue of curve labeling. The group expects to write a paper on this problem in the future.
- **2D self-avoiding random walk with high winding number.** The group considered a setup in which one looks at the law of a length n self-avoiding random walk γ in $\mathbb{Z}^2$, started at $(0,0)$ and tilted by $\exp(\eta W(\gamma))$, where $\eta > 0$ and $W(\gamma)$ is the winding number of γ around $(0,0)$. This should result in a spiral shaped trajectory, and one hopes that at the edge an appropriate scaling limit would yield the parabolic Airy line ensemble (as a resampling in terms of non-intersecting Brownian bridges in the limit seems plausible). The group tried changing the setup to be of a Schramm-Loewner evolution (SLE, the putative scaling limit of SAW) tilted by the winding number, as this might make more tools available. They then did a brief literature search and found that there is a variant of SLE that has a in-built parameter controlling its "spiralling", which seemed promising as it would have conformal covariance properties. However, further reading suggested that the spiralling produced is logarithmic, i.e., each spiral had a radius which is a constant factor of the previous layer, which suggests that the limit, if it exists, may not be the Airy line ensemble (where the gap is a polynomially decreasing function). The group concluded that the original tilted SAW model may be the most promising one for the phenomenon to occur.
- **Classification of Gibbsian Line Ensembles.** This group studied whether a Brownian Gibbs line ensemble that becomes stationary after a deterministic shift must have a quadratic or linear shape. The group developed a comparison argument based on the local behavior of the ensemble on two short intervals. This approach appears to yield the desired classification if the influence of the lower boundary vanishes when sufficiently many curves are considered. Establishing this boundary estimate rigorously remains the main technical challenge. The group also discussed possible extensions to nonstationary ensembles, curves with different shapes, ensembles on half-lines or bounded intervals, and more general interacting line ensembles.
- **Line ensemble for heavy tailed models.** This group considered directed last passage percolation (LPP) in an $n \times n$ box, where the weights satisfy $\mathbb{P}(X > t) = t^{-\alpha}$ for $t > 1$, and $\alpha \in [2, 5]$. A heuristic of Flory then says that the resulting exponents for the transversal and height fluctuations depend on α as $(1+\alpha)/(2\alpha - 1)$ and $3/(2\alpha - 1)$, respectively, as discussed in https://arxiv.org/html/2411.08018v3. The group did a back-of-the-envelope calculation that showed that if those Flory scaling exponents are correct, then the spatial process should have Holder regularity just below $3/(1+\alpha)$, which is indeed Brownian for $\alpha = 5$ and Lipschitz for $\alpha = 2$. So the group guessed that this spatial process perhaps should look locally like fractional Brownian motion with the corresponding Hurst exponent.

  With a leap of faith, they then guessed that perhaps this scaling limit may also be the limit of other types of "long-range" LPP models with spatial correlations that decay at infinity but very slowly. In other words, there may be some relationship between LPP with heavy-tailed iid weights and LPP with bounded non-iid weights but with large spatial correlations. This then led to the question of looking at fractional Brownian LPP and trying to compute its melon transform to see if that is a nice object or not. This may be a more tractable problem.

- **LPP on GFF.** This group studied last-passage percolation in heavy-tailed and discrete Gaussian Free Field environments, motivated by the preprint https://arxiv.org/abs/2411.08018 The main objective was to identify a method for proving upper bounds that match the known lower bounds. One possible approach is to reverse the geometric construction used for the lower bound and show that a path with an excessively large passage time would necessarily miss important contributions from smaller scales. The group also examined the critical heavy-tailed case and found that existing methods do not extend directly. Further progress may require new insight into geodesic geometry or a reformulation involving entropy last-passage percolation.
- **Dynamics of the Airy line ensemble.** The group discussed various proposals to put a dynamic on the Airy line ensemble which kept its distribution invariant. One dynamic was the natural evolution of the line ensemble in the directed landscape, i.e., by considering larger heights/times in the directed landscape and rescaling (or by considering a prelimiting analogue such as the polynuclear growth model's temporal evolution). Other proposals included evolving the noise underlying the polynuclear growth model directly, or evolving the Brownian motion entries in a Hermitian random matrix according to an OU process (thus inducing a dynamic on Dyson Brownian motion). A question of interest related or relevant to most of these proposals was how the directed landscape (or noise) was related to the Airy line ensemble, e.g., if the noise is updated in some way, what can one say about how the line ensemble changes, or vice versa? A concrete version of the same would be to update the LPP line ensemble associated to a solvable weight distribution (such as geometric) according to its Gibbs property and see how it changes the original i.i.d. environment. The group explored various computed examples of this process to gain some intuition, and noted that an extended invariance identity equating LPP values in the original i.i.d. model and in the associated line ensemble may give a way to access which vertex weights in the i.i.d. environment are modified by the Gibbs resample. This seemed promising but it was not clear how tractble a description it is.
- **Toda CFT.** This group examined whether probabilistic Toda CFT can be regarded as a two-dimensional analogue of a Gibbsian line ensemble, working from the definition to record the deterministic-domain DLR resampling property and to see what does and does not transfer from the Brownian Gibbs setting.

Summary

The workshop brought together researchers studying Gibbsian line ensembles from a variety of perspectives, including integrable probability, stochastic partial differential equations (SPDEs), and statistical mechanics. It helped connect these different viewpoints, stimulated new collaborations, and identified several promising research directions and open problems. The organizers are already involved in several projects related to ideas discussed during the workshop and plan to continue actively developing them.

The organizers are deeply grateful to the AIM staff for their extraordinary support, hospitality, and encouragement, all of which contributed greatly to the success of the workshop.