Global Langlands correspondence

organized by
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Workshop Summary

Introduction

The goal of the workshop was to give the exposition of Vincent Lafforgue's proof of the Automorphic $\to$ Galois direction in the Langlands correspondence over function fields.

The Langlands program emerged as an organizing principle in the theory of automorphic forms. It requires two pieces of data as an input: a global field F and a (connected) reductive group G over F. To such G, one can attach the space $\mathcal{A}(G)$ of automorphic forms of G. The Langlands correspondence aims to describe this space by a spectral datum closely related to arithmetic, i.e. (semisimple) continuous homomorphisms φ from the conjectural Langlands group $\mathcal{L}_F$ (an extension of the Weil group W_F of F by a locally compact Lie group) to the complex L-group ${}^LG = \hat{G} \rtimes \mathrm{Gal}(\overline{F}/F)$. If F is a global function field of characteristic $p > 0$, one may replace φ by semisimple continuous ℓ-adic representations W_F to $\overline{\mathbb{Q}}_\ell$ version of ${}^LG$. Such a correspondence, once available, will be a powerful tool in solving various arithmetic as well as representation-theoretic questions. The Langlands program has been an active research area over the past few decades. Previously, only the case of $G = \mathrm{GL}_n$ over function fields was known ($n = 2$ by V. Drinfeld and $n > 2$ by L. Lafforgue). V. Lafforgue's method is completely different from all the previous approaches. He ingeniously combined two ideas of V. Drinfeld (the moduli space of Shtukas and the geometric Langlands program) to construct the so-called *excursion operators* that are key for the sought-for spectral decomposition.

Workshop talks

During this one week workshop, we see an essentially complete proof of V. Lafforgue's results with the latest technical and conceptual improvements and simplifications, some of which are not publicly available yet (this includes Varshavsky's simplification of the proof of the $S = T$ theorem, a main technical result of Lafforgue's paper, and Xue's study of the cuspidal cohomology). Therefore, the logic of the exposition was somewhat different from V. Lafforgue's paper.

It proceeded in the following order:

Monday morning:

(1) Overview (Xinwen Zhu)

(2) Background on the Affine Grassmannian and Geometric Satake (Sophie Morel)

Monday afternoon:

(2') Further background on the Affine Grassmannian and Geometric Satake (Sophie Morel)

Tuesday morning:

(3) The moduli space of shtukas (Zhiwei Yun)

(4) Action of partial Frobenius morphisms (Benot Stroh)

Wednesday morning:

(5) The S-operator (Dennis Gaitsgory)

(6) The Cayley-Hamilton idenity for the S-operator, a.k.a. Eichler-Shimura (Yifei Zhao)

(7) The fundamental $S = T$ identity (Yakov Varshavsky)

Wednesday afternoon:

(7') The fundamental $S = T$ identity continued (Yakov Varshavsky)

Thursday morning:

(8) Hecke-finite part of the cohomology of shtukas and construction of the system of functors $H_I : \mathrm{Rep}(\check{G}^I) \to \mathrm{Rep}(\mathrm{Gal}^I)$ (Dennis Gaitsgory)

(9) Proof of Drinfeld's lemma (Benot Stroh)

Friday morning:

(10) Construction of semi-simple homomorphisms $\mathrm{Gal} \to \check{G}$ starting from a system of functors $H_I : \mathrm{Rep}(\check{G}^I) \to \mathrm{Rep}(\mathrm{Gal}^I)$ (Xinwen Zhu)

(11) Verification of continuity, non-ramification away from the level, and the Hecke eigen-condition of the resulting homomorphisms $\mathrm{Gal} \to \check{G}$ (Sophie Morel)

In addition we had several research talks, which were either further developments/applications of Lafforgue's results, or closely related:

(1) Wei Zhang, report on the paper "Shtukas and Taylor expansions of L-functions", by Z. Yun and W. Zhang;

(2) Gebhard Boeckle, report on the paper "$\check{G}$-local systems on projective curves are potentially automorphic", by G. Boeckle, M. Harris, C. Khare and J. Thorne (Wednesday afternoon);

(3) Cong Xue, report on her thesis establishing the equivalence between the Hecke-finite and cuspidal part of the cohomology of shtukas (Thursday afternoon);

(4) Sophie Morel, relation between V. Lafforgue's construction and Arthur multiplicity conjectures (Friday afternoon);

(5) Dennis Gaitsgory, report on the work of A. Genestier and V. Lafforgue on the local case of the Langlands conjecture.

Discussions

It is for sure such a big result would lead to a lot of discussions, both for better understanding the proof and for future research. Thanks to the very informal atmosphere, the small size of the group and the very nice space we had, it was easy to go off and have discussion with other workshop participants at pretty much any point. This allowed participants to discuss the workshop material in small groups or update each other on their research.

There were discussions among all participants about:

(1) The interpretation of the system of functors $H_I : \mathrm{Rep}(\check{G}^I) \to \mathrm{Rep}(\mathrm{Gal}^I)$ as quasi-coherent sheaves on $\mathrm{Hom}(\Gamma, \check{G})/\check{G}$. It turns out this interpretation gives a more conceptual understanding of Lafforgue's construction of excursion operators and some basic properties of these operators.

(2) The relationship between conjectural shape of the Hecke-finite part of the cohomology of moduli of Shtukas (through the above interpretation) and the Kottwitz conjecture of the shape of cohomology of Shimura varieties, in particular the non-tempered part.

(3) Understanding of Drinfeld's lemma.

(4) Understanding of the fundamental $S = T$ equality in terms of the formalism of cohomological correspondences, due to Y. Varshavsky.

(5) Interpretation of the Hecke-finiteness condition as cuspidality, due to C. Xue.