Graph Ramsey theory

organized by
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Workshop Summary

Overview and Workshop Focus. The workshop, sponsored by the AIM and the NSF, focused on Ramsey theory for graphs and hypergraphs, and geometric Ramsey problems. The Ramsey number $r(H)$ for a k-uniform hypergraph H is the minimum n such that any red/blue coloring of the edges of the complete k-uniform hypergraph on n vertices results in a monochromatic copy of H. Several basic problems about Ramsey numbers of graphs and hypergraphs have remained unsolved for many decades.

The purpose of this workshop was twofold: to consolidate the current knowledge on Ramsey theory, and put forth a research agenda for the future. The workshop focused on three areas: generalized Ramsey numbers for graphs, hypergraph Ramsey numbers, and geometric Ramsey problems. There were a number of survey talks representing the current state of each field; these discussed many of the important outstanding problems in the subject and the most successful modern proof techniques. On the first afternoon itself, there was an extensive problem session, led by Jacques Verstraete, which helped the organizers to select a list of problems for group work later in the week.

There were 32 participants in the workshop, from the US, Canada, Germany, the UK, and Hungary. These included some of the senior experts in the subject like Ron Graham and Vojtech Rödl as well as many young researchers and graduate students.

Summary of Talks. There were several introductory lectures. On the first day, we had talks by David Conlon and Choongbum Lee. Conlon spoke on the basic methods and results about hypergraph Ramsey numbers. These include the pigeonhole argument of Erdős and Rado and the stepping-up lemma due to Erdős and Hajnal. Lee spoke about an old problem, originally posed by Erdős and Shelah, concerning generalized Ramsey numbers, where we wish to find an edge-coloring such that every clique of a fixed order contains at least some fixed number of colors. On the second day, we had talks by Jacob Fox and András Gyárfás. Fox's talk was a thorough treatment of the many applications in Ramsey theory of the powerful technique known as Dependent Random Choice, while Gyárfás discussed a number of new problems relating the chromatic number to Ramsey theory. On the third day, we had talks from Andrzej Dudek on the Erdős-Rogers problem and Maria Chudnovsky on the Erdős-Hajnal problem. Both of these problems have been open for decades although there has been significant recent progress, particularly on the former. On the fourth day, we had a talk given by Jozef Skokan on Ramsey goodness and related results. Finally, on the last day of the workshop we had a talk by Ron Graham on a variety of old and new open problems, mainly in Euclidean Ramsey theory. Taken together, these high quality lectures gave an excellent overview of the important techniques, recent developments and future challenges in graph, hypergraph, and Euclidean Ramsey theory.

Summary of Working Groups.

The organizers selected a list of 8 open problems for further study on the second day of the workshop, and this formed the basis of group research for the rest of the week. One group worked on determining the growth rate of the hypergraph Ramsey number $r_k(k+1)$, which is the minimum n such that every 2-edge-coloring of the complete k-graph on n vertices results in a monochromatic copy of a complete k-graph on $k+1$ vertices. Partial progress on this problem was achieved. Another group considered a recent geometric problem, that of determining $g(p)$, the minimum n such that for every collection of n points in the plane in general position, and every 2-edge-coloring of the pairs of points, there is a collection of p points in convex position all of whose pairs receive the same color. There was no substantial progress to report here. A third group worked on the generalized Ramsey problem of determining $F(p,k)$, the minimum n such that every k-edge-coloring of K_n results in a copy of K_p whose edges receive at most $p-2$ colors. Some interesting observations about constructions for small p were made, giving improved lower bounds. Another problem considered was that of determining the ordered Ramsey number for triangles versus matchings. Some preliminary results were obtained here and we can expect more results in the future. Progress was also made on some special cases of the very general problem of determining $r(F, K_t^{(3)})$ for various 3-graphs F and large t. Finally, the problem of determining the Ramsey number of knots was considered. While the group in question quickly realised that the bounds given by Negami in his paper on Ramsey numbers of knots could be improved from a five-fold exponential to a double exponential, it seems that a new idea will be needed to push the bounds past this bottleneck.

Partial Results. The following partial results were obtained:

Ramsey number $r_k(k+1)$. The related question when we consider more than 2 colors or when we seek a monochromatic clique of size $k+2$ (instead of $k+1$) had been studied by Duffus, Lefmann, and Rödl and for these cases tower-type lower bounds had already been obtained. The goal was to obtain similar tower-type bounds for $r_k(k+1)$. By connecting this problem to an ordered Ramsey problem for tight paths, one group was able to give a tower-type lower bound for $r_k(k+1, k+2)$. This was not previously known and is very close to what was originally sought. It was achieved by using recent results of Moshkovitz and Shapira and Milans, Stolee, and West on ordered Ramsey numbers of tight paths and estimates on the size of certain posets created by an iterative procedure. The same approach was used to prove a new lower bound on the off-diagonal hypergraph Ramsey number $r_k(k+1, n)$, which is $(k-3)$-fold expoential, and one exponential off from the upper bound. Previously, it was known (Conlon, Fox, and Sudakov, in unpublished work) to be at least $(k-5)$-fold exponential by a variant of the stepping-up approach.

Edges in many triangles. A *tree of triangles* is any graph built recursively by starting with a triangle and then adding, one at a time, a vertex and two edges from this vertex that form a single new triangle with an existing edge. It is known that for any tree T of triangles, there exists k such that if a graph G has the property that every k-edge-coloring of G contains a monochromatic triangle, then G contains T. Rödl proposed the following question. For every ℓ, is there a graph G in which every ℓ vertices induce a subgraph which is a tree of triangles and yet every 2-edge-coloring of G contains a monochromatic triangle?

As a weaker question, can you find a graph with the first property and where every edge is in at least two triangles? This weaker question was solved by Verstraete during the workshop.

Ordered Ramsey problems An ordered graph on N vertices is a graph whose vertices have been labeled with $\{1, \dots, N\}$. An ordered graph G on $\{1, \dots, N\}$ contains another ordered graph H on $\{1, \dots, n\}$ if H occurs as a subgraph of G with vertices appearing in the correct order. Let $r_<(H_1, H_2)$ denote the minimum N such that every red/blue-coloring of the edges of the complete graph on $\{1, \dots, N\}$ contains a monochromatic red ordered copy of H_1 or a monochromatic blue ordered copy of H_2. This group worked on estimating the Ramsey number $r_<(K_3, M)$, where M is an ordered matching. Conlon, Fox, Lee, and Sudakov asked if there is $\epsilon > 0$ such that for every matching on n vertices this ordered Ramsey number is $O(n^{2-\epsilon})$. For certain ordered matchings, the group determined this Ramsey number precisely. Maybe the most exciting development here is that they came up with an analogue of Ramsey goodness for ordered Ramsey numbers, and showed that certain matchings are 3-good.

Ramsey number of F versus a clique. The question of determining the Ramsey number of the tight path versus a complete graph was considered by a small group. Preliminary ideas for a geometric construction achieving the correct order of magnitude for k-graphs when $k \geq 4$ were obtained. Further progress is certainly expected. The question of determining the Ramsey number of the tight cycle versus a complete graph was considered by another small group. Here substantial progress was made on the last day of the workshop with reasonable exponential upper and lower bounds obtained.

Concluding remarks. Based on the feedback from participants (especially the younger researchers) and progress on problems, we think the workshop was a success. It is expected that a number of new collaborations may result from the workshop and at least a few papers should grow out of the problems considered.