Groups of dynamical origin

organized by
Rostislav Grigorchuk, Constantine Medynets, and Dmytro Savchuk

Workshop Summary

Overview

- **Organizers:** Rostislav Grigorchuk (Texas A&M University), Constantine Medynets (United States Naval Academy), and Dmytro Savchuk (University of South Florida)
- **Dates:** June 3 – June 7, 2024

This workshop, sponsored by AIM and NSF, brought together researchers interested in group theory, number theory, dynamical systems, and C^*-algebras with interests revolving around topological full groups, groups acting on rooted trees, orbit equivalence theory, automata groups, symbolic and tiling dynamics, and arboreal representations of Galois groups to discuss recent developments in the subject with the focus on ample-like groups, their actions and algebraic properties.

The main topics for the workshop were:

- The interplay between topological dynamics, group theory, and number theory,
- Self-similarity of groups and their actions,
- Orbit equivalence theory of group Actions,
- Growth of groups and graphs

This event was run as an AIM-style workshop. Participants were invited to suggest open problems and questions before the workshop began. These included a variety of specific problems on which there is hope of making some progress in the near future as well as more ambitious problems which may influence the future activity of the field. Lectures at the workshop were focused on familiarizing the participants with the background material leading up to specific problems and the speakers were selected in accordance to the problems and questions proposed by the workshop participants.

Workshop Speakers

(1) Jamie Juul, Colorado State University, Title of the talk: *Arboreal Representations of Galois Groups.*
(2) Volodymyr Nekrashevych, Texas A&M University, Title of the talk: *The Groups of Dynamical Origin.*
(3) Natalie Priebe Frank, Vassar College, Title of the talk: *Tiling Spaces.*
(4) Nicolás Matte-Bon, Université Lyon 1, Title of the talk: *Amenability of Topological Full Groups.*
(5) Rachel Skipper, University of Utah, Title of the talk: *Maximal Subgroups of Thompson's Groups.*
(6) Collin Bleak, University of St Andrews, Title of the talk: *Hyperbolic groups satisfy the Boone-Higman conjecture.*

(7) Yaroslav Vorobets, Texas A&M University, Title of the talk: *Overview of Topological Full Group.*
(8) Alina Vdovina, The City College of New York (CUNY), Title of the talk: *On Multi-dimensional Grigrochuk Groups.*
(9) Ophelia Adams, University of Rochester, Title of the talk: *Arboreal Representations of Galois Groups.*
(10) Sebastian Barbieri, Universidad de Santiago de Chile, Title of the talk: *On Multidimensional Sofic Shifts.*

Working Groups

Each day the workshop participants broke into smaller groups to work on a particular problem. Some groups stayed active for the duration of the workshop, while others shifted their focus to different problems. In what follows we present a subset of problems the participants of the workshop worked on.

The Fixed Point Proportion of Dynamically Exceptional Polynomials (Moderator Santiago Radi).

Let G be a group acting on a infinite rooted tree, and denote G_n the action of G on the first n levels of the tree. We define the **fixed-point proportion** of G as

$$FPP(G) = \lim_{n \to +\infty} \frac{\# \{g \in G_n : \ g \text{ fixes at least one element on level } n\}}{\# G_n}.$$

Let k be a field, $f \in k(z)$ of degree at least 2, and $t \in k$. Define $k_n = k(f^{-n}(t))$, $G_n = Gal(k_n/k(t))$ and $G_\infty = \varprojlim G_n$. The group G_n acts on $f^{-n}(t)$ via the natural action of the Galois group, so G_∞ acts on the infinite rooted tree of the n-th preimages, $n = 1, 2, \ldots$ of t via f.

In particular, if $k = \mathbb{C}(t)$ with t transcendental over $\mathbb{C}$, G_∞ is isomorphic to the closure of the iterated monodromy group of f, denoted $IMG(f)$ (see [**?**, Proposition 6.4.2]). Results in this direction can be found in [Jones$_I terated_m onodromy_g roups$], $whereitisprovedthatfornondynamically$
$= 0$.

Questions:

(1) What can we say about the fixed-point proportion of groups acting on rooted trees?
(2) Fixing the tree T, can we find a family of groups such that its fixed-point proportion is nonzero and its Hausdorff dimension converges to 1?
(3) What can we say about the fixed-point proportion of Galois groups?
(4) What can we say about the fixed-point proportion of $IMG(f)$ of complex rational functions?
(5) What can we say about the fixed-point proportion of $IMG(f)$ of complex polynomials that are not dynamically exceptional?

A progress on this problem has been made and a preprint [radi:fpp] authored by Santiago Radi, one of the group participants, is available per request and will be posted and submitted soon.

Analogues of Paths for Calculating Profinite Iterated Monodromy Groups (Moderator Ophelia Adams)

Let k be a field, $f \in k(z)$ of degree at least 2, and t be a transcendental number. Define $k_\infty = k\left(\cup_{n\geq 0} f^{-n}(t)\right)$, $L = k_\infty \cap \overline{k}$, $G_\infty = Gal(k_\infty/k(t))$ and $G_\infty^{geom} = Gal(k_\infty/L(t))$. This is the geometric profinite iterated monodromy group associated to the rational function f.

When $k = \mathbb{C}$, then $G_\infty^{geom} = \overline{IMG(f)}$ (topological closure within the profinite group of all tree automorphisms) and for the iterated monodromy group of a continuous covering map, there is an explicit way to construct it by choosing (specific) paths between the chosen base point z_0 and points in $f^{-1}(z_0)$ (see [**?**, Proposition 6.4.2]). This allows one to explicitly describe the generators by wreath recursions within the automorphism group of the tree, giving more immediate access to its structure. We lack the tools to do so algebraically. However, Richard Pink showed that, in some cases (quadratic PCF), these groups can be determined algebraically over arbitrary fields (characteristic not 2) without so directly using paths and topological methods.

Question: Can we algebraically construct "specific paths" to describe G_∞^{geom} by more explicit wreath recursions? More precisely, given $z \in f^{-1}(t)$, can we find explicit endomorphisms of K_∞, perhaps recursively described, sending t to z with nice properties?

Presently, we can only get wreath recursions up to conjugacy by algebraic methods, and Pink shows that for quadratic PCF polynomials these choices do not matter. We cannot expect to easily obtain "specific" paths in general, because this would be significant progress toward calculating the étale fundamental group of punctured $\mathbb{P}^1_{\overline{K}}$ in purely algebraic terms, a problem which has been open for several decades. This is why we might expect, for example, a recursive description at best, something highly dependent on the structure of K_∞.

(Outer) (Tree) Automorphism Groups of (Profinite) Iterated Monodromy Groups (Ophelia Adams)

The arithmetic iterated monodromy group contains, as a normal subgroup, the geometric iterated monodromy group, and so it induces automorphisms (by conjugation) of the geometric iterated monodromy group. This naturally suggests a series of related questions, where G^{geom} denotes the (profinite) geometric iterated monodromy group.

Questions:

- What is the automorphism group of G^{geom}?
- Which automorphisms of G^{geom} are induced by tree automorphisms?
- What is the outer automorphism group of G^{geom}?
- Which outer automorphisms of G^{geom} are induced by tree automorphisms?

Broadly speaking, it is the outer automorphism group which is most interesting to those in arithmetic dynamics.

Richard Pink calculated the tree automorphism and outer automorphism groups associated to post-critically finite quadratic polynomials and applied this analysis to determine or constrain the associated constant field extensions, arithmetically interesting objects. For example, in the strictly pre-periodic non-Chebyshev cases, the outer automorphism group turns out to be 2-torsion, hence the absolute Galois group's image in it has order at most 4. Pink's main results are somewhat stronger than this, but a version of this fact for rational maps or higher degree polynomials would be quite interesting; simply distinguishing finite from infinite image in the outer automorphism group, especially with good control over the size in the finite case, would have interesting applications in dynamics.

It is worth pointing out that Pink's calculations rest on a semirigidity result for profinite IMGs: he shows that a subgroup of the automorphism group of the binary tree with generators satisfying certain wreath recursions up to conjugacy is conjugate, within the automorphism group, to a particular "model group". This allows to avoid topological methods over $\mathbb{C}$. He then calculates the automorphism and outer automorphism groups from the more explicitly defined model group.

This working group focused mostly on the contributions to the outer automorphism group from the Galois action; the presence of a totally ramified fixed point (∞) strongly constrains the nature of these outer automorphisms (both group theoretically and number theoretically) and allowed us to reduce our main questions to a single family of conjugacy problems around the associated odometer.

Ophelia Adams and Trevor Hyde had independently began work on these questions prior to the workshop, and have merged their work into a collaborative project through AIM. At the workshop, the group made very good progress on these questions for unicritical PCF polynomials and intend to continue and finish this work in the near future.

Orbit Equivalence Theory for Groups Acting on the Boundary of Rooted Trees (Moderator Maria Cortez)

Orbit equivalence theory tries to understand the topological structure of group orbits up to orbit equivalence. Let G, H two groups acting on Cantor sets X and Y. We say that G and H are **orbit equivalent** if there is a homeomorphism from X to Y that sends orbits of G onto orbits of H. The classification of minimal actions of the group of integers up to orbit equivalence has been obtained in the seminal works by Giordano, Putnam, and Skau [giordano$_{p}s:full_{g}roups_{o}f_{c}antor_{m}inimal_{s}ystems99].Theyalsomadesomeinroadsintotheclassificationofminimal\mathbb{Z}^n$-actions [MR2563761] where it was shown that every minimal $\mathbb{Z}^n$ is orbit equivalent to a $\mathbb{Z}$-action. In particular, $(X, \mathbb{Z}^2)$ and $(Y, \mathbb{Z})$ are orbit equivalent. However, finding an orbit equivalence map between both systems presents a significant challenge.

The focus of this group was to try to extend the orbit equivalence results to other classes of groups, in particular, in the spirit of this workshop, to the class of iterated monodromy groups.

Question 0.1. *Find an explicit orbit equivalence map between to orbit equivalent odometers.*

Question 0.2. *Suppose that Γ is a group acting on the boundary of a rooted tree, e.g. the iterated monodromy group of a continuous map. Is there a $\mathbb{Z}$-action with the same orbits? Find an explicit homeomorphism implementing the orbit equivalence.*

The structure of Topological Full Groups (Moderators Constantine Medynets and Volodymyr Nekrashevych)

There were several work groups interested in the study of topological full groups of minimal symbolic systems.

Algebraic Characterization of the Topological Entropy. Let A be a finite alphabet, A^* the set of finite words in A and $\mathcal{F} \subseteq A^*$ a set of forbidden patterns that contains the set $\{aa : a \in A\}$. Let

$$S_{\mathcal{F}} = \left\{x \in A^{\mathbb{Z}} :\ x \text{ contains no subword in } \mathcal{F}\right\}.$$

Define $\varphi_a \in Homeo(S_{\mathcal{F}})$ as follows:

$$\varphi_a : \begin{cases} \text{shift } x \text{ to the left,} & \text{if } x(1) = a, \\ \text{shift } x \text{ to the right,} & \text{if } x(0) = a \\ x, & \text{otherwise} \end{cases}$$

and $G_{\mathcal{F}} = \langle \varphi_a : a \in A \rangle$.

Recall that the **topological entropy** is defined as

$$\mathcal{H}_{\mathcal{F}} := \lim_{n \to +\infty} \frac{\log\left(\#\text{ of words of length } n \text{ that appear in some } x \in S_{\mathcal{F}}\ \right)}{n}.$$

and $F(G_{\mathcal{F}})$ is the **full topological group**. A group G is called **residually finite** if

$$\bigcap_{H \leq G : [G:H] < \infty} H = 1$$

Question 0.3 (Medynets)**.** (1) *Describe relations in $G_{\mathcal{F}}$.*
(2) *Can $\mathcal{H}_{\mathcal{F}}$ be algebraically interpreted as an invariant inside $G_{\mathcal{F}}$?*
(3) *If $\mathcal{H}_{\mathcal{F}} = 0$, does either $G_{\mathcal{F}}$ of $F(G_{\mathcal{F}})$ not contain free subgroups?*
(4) *When is $G_{\mathcal{F}}$ residually finite?*
(5) *When does $G_{\mathcal{F}}$ contains a finitely generated subgroup of intermediate growth?*

The discussions held by the group that worked on this topic concentrated their efforts on the second question. Namely, on whether the topological entropy can be recovered abstractly from an algebraic description of a topological full group.

It is worth mentioning that for Cantor minimal systems, it is well-known that the topological full group is a full invariant of flip conjugacy [giordano$_p s : full_g roups_o f_c antor_m inimal_s ystems99]and$

An old result of Fremlin, which is also known as the Rubin reconstruction theorem, states that any isomorphism between groups of automorphisms of complete Boolean algebras is always generated by an isomorphism of the underlying algebras. In [Medynets2011] it is shown that for general Cantor systems (not necessarily minimal!) satisfying a very mild technical hypothesis, any isomorphism between topological full groups is induced by a homeomorphism of the Cantor sets, which, in its turn, implements an orbit equivalence. A key part of this is finding algebraic criteria to abstractly recover the boolean algebra of clopen sets from local subgroups, which were already given a complete algebraic characterization in the original proof of Dye's result. Finally, it can be shown that one (and hence also the other) of the cocycles associated to the produced homeomorphism is continuous, and it follows then from a theorem of Boyle that the underlying systems are flip conjugate.

Putting all of the aforementioned ingredients together, an abstract model for the entropy may be recovered from the topological full group itself (at least in the case where it comes from a Cantor minimal system).

Defining relations for subgroups of full group of a shift. Let A be a finite alphabet and Ω be a 2-sided subshift, i.e., a closed subset of $A^{\mathbb{Z}}$ that is invariant under the shift map σ. An element $g \in Homeo(\Omega)$ belong to the full group of the shift ($F(\Omega)$) if each point in Ω has an open neighborhood U such that $g|_U = (\sigma^n)|_U$ for some $n \in \mathbb{Z}$ depending on U is equal to resembles a power of the shift map.

Question 0.4 (Nekrashevych)**.** *Is it true that for all finitely generated subgroup $H \leq F(\Omega)$, H is either elementary amenable or for any K finitely presented subgroup such that $K \twoheadrightarrow H$, K contains a free subgroup?*

Two inspiring results come from [MR3061134], [MR3395261] and [nekrashevych:self-similar$_g roups_a nd_t opological_d imension$] :

(1) There exists G such that the Grigorchuk group of intermediate growth embeds as a subgroup of G but the Grigorchuk group does not have a group K in the conditions of the question.
(2) The same as the previous item holds if we replace the Grigorchuk group but the iterated monodromy group of a expanding covering map $f \colon J \to J$ with $\dim(J) = 1$.

On the Commutator Width of Transformation Groups (Moderator Ashley Johnson)

Thompson's group F is the group of orientation-preserving piecewise linear automorphisms of the unit interval $[0, 1]$ for which all slopes are powers of 2 and breakpoints lie in the dyadic rationals. Thompson's group T acts in a similar way but on S^1, and Thompson's group V on the Cantor set. For a full introduction to F, T and V, see the Cannon, Floyd, and Parry notes [MR1426438].

Let G be one of Thompson's groups F, T or V, or a topological full group, and let $g \in G$ be an arbitrary element. Let $\rho_1(g)$ be the commutator width of g and $\rho_2(g)$ the minimum number of involutions in a factorization of g.

(1) Prove that ρ_1 and ρ_2 are bounded.
(2) Prove that $\rho_1 = 1$ and $\rho_2 = 3$.
(3) Is there a finitely presented simple group with commutator width greater or equal to 2?

A result of Dennis and Vaserstein [MR1010982] applied to F gives that the commutator width of F', the commutator subgroup of F, is at most 2. It is conjectured by Collin Bleak that the commutator width of F' and V are both 1, and that the commutator width of T is 2. We believe there are few examples (perhaps only one?) of infinite simple groups with commutator width greater than 1, and none as naturally occurring as T. In particular, a candidate for an element in T that is not a commutator was proposed to be $\rho_{1/2}$, the half rotation of S^1.

Entropy of Subshifts of Finite Type (Moderator Sebastián Barbieri)

Let G be a finitely generated infinite amenable group, A be a finite alphabet and define

$$\mathcal{F} = \{f \colon B \to A \mid B \subseteq G \text{ finite}\}$$

a finite set of forbidden patterns and

$$S_{G,\mathcal{F}} = \left\{x \in A^G : x \text{ contains no forbidden subpatterns}\right\}.$$

G acts on $S_{G,\mathcal{F}}$ on both sides as the regular representation, namely, permuting the coordinates of the sequences by right or left translation. Choose $\{F_n\}_{n\in\mathbb{N}}$ a Følner sequence for G and define

$$\mathcal{H}_{G,\mathcal{F}} = \lim_{n\to+\infty} \frac{1}{n} \log\left(\#\left\{f : F_n \to A : \; f \text{ appears in some } x \in S_{G,\mathcal{F},A}\right\}\right).$$

It is a fact that $\mathcal{H}_{G,\mathcal{F}}$ does not depend on the selected Følner sequence. Define also

$$E_G = \{\mathcal{H}_{G,\mathcal{F}} : \mathcal{F} \text{ is a finite set of forbidden subwords}\}.$$

It is known (see [Lind$_M arcus_1 995]and[MR2680402])thatE_{\mathbb{Z}} = \{q\log(\lambda) : q \in \mathbb{Q}^+, \text{ and } \lambda \text{ is a Perron e}$
$\{r \in \mathbb{R}^+ : \; r \text{ is a right-recursively enumerable number}\}, whereaPerroneigenvalueistheeigenvaluegivenby$
$Frobeniustheoremofasquarematrixwithnon-negativeentriessuchthatforsomepowerofthematrix, theen$
$\mathbb{Q}$.

Question 0.5 (Barbieri)**.** (1) *Given a recursively presented and finitely generated infinite amenable group G, what is E_G?*
(2) *Does there exist a recursively presented and finitely generated infinite amenable group G such that $E_{\mathbb{Z}} \subsetneq E_G \subsetneq E_{\mathbb{Z}^2}$?*

Results known: if G is a finitely generated branch group with decidable word problem, then $E_G = E_{\mathbb{Z}^2}$. See [MR4303334].

Growth Functions of Repetitive Graphs (Moderator Nicolas Matte-Bon)

A graph Γ is called **repetitive** if for all $r > 0$ there exists $d > 0$ such that for all $x, y \in \Gamma$, there exists an isomorphic copy of $B_r(x)$ (the ball of radius r centered at x) at distance less or equal to d from y.

Consider two kind of growths functions for Γ: pick $x_0 \in \Gamma$ and define $f_{x_0}(n) = \#B_n(x_0)$ and $f(x) = \max\{f_x(n) : x \in \Gamma\}$

Question 0.6 (Matte-Bon)**.** *What functions can be realized as growth functions of repetitive graphs?*

Simplicity of Nekrashevych algebras of contracting self-similar groups (Moderator Benjamin Steinberg)

Nekrashevych associated a C^*-algebra to each self-similar group action [NekCrelle09], which is in fact the groupoid C^*-algebra of an ample groupoid. He also studied [NekGrowth16] the complex $*$-algebra of this groupoid. Groupoids associated to faithful self-similar group actions are minimal and effective, but rarely Hausdorff. Therefore, they form a nice family of test examples in trying to understand when algebras associated to non-Hausdorff groupoids are simple. An important open question is whether the reduced C^*-algebra of an ample groupoid is simple if and only if the complex $*$-algebra is simple. It is known that simplicity of the compelx $*$-algebra is necessary for simplicity of the C^*-algebra [CESS19].

In [SS23], Steinberg and Szakács provided an algorithm to decide, given the nucleus as input, simplicity of the complex $*$-algebra associated to a contracting self-similar group. It was shown in [CESS19] that the C^*-algebra associated to the Grigorhcuk group is simple.

Gardella, Nekrashevych, Steinberg and Vdovina proved during the meeting that the C^*-algebra associated to a contracting group is indeed simple if and only if the complex $*$-algebra is simple. It is well known that the obstruction to simplicity of the C^*-algebra and the $*$-algebra depends on triviality of the essential ideal [CESS19,NekGrowth16]. The key idea is to show that if the C^*-algebraic ideal is non-empty, then there is a sequence of elements of this ideal, bounded away from 0, that accumulates in the span of the nucleus, which is a finite dimensional subspace contained in the compelex $*$-algebra. This leads to the existence of a nonzero element in the algebraic essential ideal.

Articles That Benefited from the Workshop

- Sasntiago Radi, *A Family of Level-Transitive Groups with Positive Fixed-Point Proportion and Hausdorff Dimension*, 2024.
- Artem Dudko, Constantine Medynets, *On Characters of Topological Full Groups of Minimal $\mathbb{Z}$-systems*, 2024, in preparation.
- Delaram Kahrobaei, Arsalan Akram Malik, Dmytro Savchuk, *Contracting Self-similar Groups in Group-Based Cryptography*, https://arxiv.org/abs/2408.14355, 2024.