Higher Du Bois and higher rational singularities

organized by
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Workshop Summary

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In this document, we summarize the focus of each of the working groups during the AIM workshop, "Higher Du Bois and higher rational singularities", which took place from October 27, 2024, to November 1, 2024.

Group 1: Relative Du Bois Complex. The essence of the problem is that there is a need for the construction of a complex for families that plays a similar role to the Du Bois complex for reduced schemes.

Kovács has constructed objects for families that behave similarly to the associated graded complexes of the Du Bois complex and Kovács and Taji constructed objects for families over smooth curves that behave similarly to the full Du Bois complex. Both of these constructions were motivated by explicit applications and thus they may or may not be the "right" objects for other applications.

In particular, one does not expect that these objects commute with base change. However, it is likely that no object fulfills all desired properties and commutes with base change because that would have very strong deformation theoretic consequences, which are not expected to hold without further assumptions.

At the workshop, the group first explored the question of what additional condition(s) would imply that the Kovács-Taji complex commutes with base change, but there were no promising leads. The group did determine that some of the first ideas that one may have do not lead to a satisfactory answer.

Later in the workshop, the group investigated a different approach: defining the relative Du Bois complex using h-differentials or other direct constructions (the Kovács-Taji method constructs the complex via a recursive process). Some of the ideas were promising and may lead to a solution in the future. The group felt that some progress had been made, and several members indicated their interest in continuing to work on the problem.

Group 2: Higher F-splittings. In their preprint Kawakami and Witaszek introduced the notion of a variety X over a field of positive characteristic having m-F-injective singularities for $0 \leq m \leq \dim X$. Motivated by their definition, one is naturally led to the following definition (due to Witaszek). If X is a normal variety and D is a reduced divisor, the pair (X, D) is said to be globally m-F-split if the logarithmic Cartier map

$$C_X \colon Z\Omega_X^{[i]}(\log D) \to \Omega_X^{[i]}(\log D)$$

is split surjective for all $i \leq m$.

When X is smooth, projective, and n-dimensional, this notion interpolates between X being F-split (which agrees with 0-F-split as defined above) and X being F-liftable (meaning $(n-1)$- F-split). The global geometry of an F-liftable variety is very constrained: a conjecture by Achinger, Witaszek, and Zdanowicz predicts that, after an étale cover, the Albanese morphism $X \to \mathrm{Alb}X$ is a toric fibration over an ordinary abelian variety. The group which worked on this problem tried to exhibit varieties/pairs which are m-F-liftable for some $m < n-1$. The group considered (cones over) products $Y = S \times E$, where E is an ordinary elliptic curve and S a K3 surface violating Bott Vanishing. These turned out to not even be 1-F-split, and actually seemed to suggest that, at least for cones over ordinary CYs, 1-F-split implies $(n-1)$-F-split. The next candidate, which the group has continued working on after the workshop, is the pair $(\mathbf{P}^3, D)$ where D is the quadric $(xy - wz = 0)$. The direct way of approaching the problem (i.e. explicitly exhibiting a splitting) seems computationally heavy, and the question was raised whether this kind of problem is amenable to Macaulay2 techniques.

Group 3: Examples of Mixed Hodge structures on cubics

The main aim of the problem is to compute the Hodge-Du Bois numbers for explicit examples of singular varieties with higher Du Bois/rational singularities. A natural testing ground is cubic hypersurfaces. The group began by reviewing the situation for cubic threefolds with isolated singularities, whose Hodge-Du Bois numbers were computed by two members of the group (Marquand-Viktorova) before the workshop. Here, the Hodge-Du Bois numbers are determined by two local invariants of the singularities of X (the Du Bois invariant and Link invariants), and a global invariant, the defect $\sigma(X)$.

Next, the group discussed the situation for a Calabi-Yau threefold with rational singularities, and assuming the existence of a curve of singularities. The group investigated what should be the correct replacement for the Du Bois and link invariants in the non-isolated case.

After that, the group discussed the situation for cubic fourfolds with non-isolated singularities. The goal was to interpret the GIT stability of a singular cubic fourfold in terms of the 1-rational/1-Du Bois conditions on the singularities. One can show that a cubic fourfold with 1-rational singularities in fact has only ADE singularities and hence is GIT stable. The more interesting situation is again a cubic fourfold which is singular along a curve. If the 1-Du Bois condition is imposed, the outer edges of the Hodge-Du Bois diamond are determined, and it seems likely that the defect can be computed in terms of the genus of the curve. It seems probable that one can show that a cubic fourfold with 1-Du Bois singularities is GIT semi-stable, and compute the remaining Hodge-Du Bois numbers, possibly by cutting with hyperplane sections as in the work of Park-Popa and using results for cubic threefolds.

Group 4: Definition of Higher singularities in non-local complete intersection setting. This problem concerned one of the three main goals of the workshop. To say that a variety Z has k-Du Bois singularities should have some implications related to those which are known in the local complete intersection (LCI) setting. It is understood that in the non-LCI setting, one should not compare Kähler differentials with the Du Bois complexes, and so one way to find a definition in general would be to find the right comparison object to replace the Kähler differentials.

After some discussion, it seemed that the following property is one of the most desirable: for flat projective morphisms, where fibers are assumed to be k-Du Bois, we would want the higher direct images of relative differentials (for the right notion of differentials and indices thereof) to be locally free and commute with arbitrary base change. This is known to hold in the LCI case by the work of Friedman and Laza.

Because it is unclear what should be assumed for the "right notion" of higher singularities outside the LCI setting, the group aimed to come up with a construction of a projective flat family over the affine line that has many of the properties that are contested and test the local freeness property mentioned above. In more detail, the group aimed to create a projective flat (partial) smoothing of a variety with many "desirable" properties and test the local freeness of higher direct images of relative Kähler differentials. The group tried this explicit calculation using Macaulay 2 and even though the programs did not terminate during the workshop, further attempts to complete this calculation can be undertaken and the property may be tested.

A completely different approach to finding the right notions was undertaken as well, using as motivation the relation with the Hodge module structure on local cohomology in the LCI setting. This group attempted to study filtrations on the local cohomology modules as this may lead to another way to define higher singularities. Morally, finding the right filtration to compare the Hodge filtration with should correspond (by taking the graded de Rham functor) with finding the right replacement of Kähler differentials. The group had some ideas, but was not able to show that these guesses had the desired properties during the workshop.

Group 5: Characteristic Class Interpretation of Higher Singularities. The goal of this problem is a generalization of a similar result proved in the hypersurface case by Maxim-Saito-Yang via spectral Hirzebruch classes. The motivation comes from the fact that such a characterization may provide an alternative for defining higher notions of rational and Du Bois singularities beyond the LCI case.

During the work sessions, this group explored potential candidates for characteristic classes of LCI varieties that could be employed to solve the above problem. For instance, one could use the Verdier specialization, or the vanishing cycles of an auxiliary hypersurface defined from locally defining equations (which also appears in the definition of the minimal exponent) to define notions of spectral classes for lci varieties. Whatever its definition, this class should be a global homological version of the Hodge spectrum. For instance, if the LCI variety has only an isolated singularity, such a spectral class should reduce to the Hodge spectrum of the corresponding Milnor cohomology (several versions of which are also available in the literature). It should have a geometric meaning, for example, one should be able to relate to results presented by Matt Kerr during the workshop, about the cohomology and mixed Hodge structures of fibers of smoothable LCI singularities.

Group 6: Singularities of an Endomorphism. This group discussed whether a singularity fixed by an endomorphism is pre-k-Du-Bois for every k. This would improve the known result that such singularities are log canonical. It could also have significant applications in other fields. The group agreed that the conjecture holds when the endomorphism lifts to a resolution of the singularity. However, their attempts to push the result further by studying the dynamics induced by the endomorphism on the space of valuations faced challenges. The group is optimistic that these challenges can be overcome outside the workshop.

Group 7: Global Behavior of the Period Map. This discussion section focused on applications of higher Du Bois and higher rational singularities to the study of the period map, with particular attention to the study of Calabi-Yau manifolds. Recent work on such manifolds has showed that there are strong connections between singular Calabi-Yau varieties Y, typically with isolated hypersurface or at least local complete intersection singularities, and which are canonical or log canonical on one hand, and the theory of 1-Du Bois or 1-rational singularities on the other. For example, in case the singularities are 1-Du Bois local complete intersection singularities, the deformations of Y are unobstructed. In the opposite direction, if the singularities are not 1-rational, then there are directions in the local deformation space of the singularities which lift to deformations of Y which, roughly speaking, do not come from deformations of a resolution. The group discussed, among other matters, the following questions:

(i) the relation between the invariants of Y and of a smoothing using the k-rational or k-Du Bois conditions;
(ii) some recent progress on compactifications of the image of the period map;
(iii) if $\widetilde{Y}$ is a resolution of Y which is not necessarily crepant (i.e. for which the canonical bundle $K_{\widetilde{Y}}$ is not necessarily trivial), are there assumptions on the singularities under which it is possible to prove a local Torelli theorem or an unobstructedness result for the deformations of $\widetilde{Y}$.

Group 8: Du Bois Vanishing Tables. This project was a combination of two problem suggestions: one by Sridhar Venkatesh about proving whether a variety Z being pre-$\lceil \frac{\dim(Z)}{2} \rceil$-Du Bois implies it is pre-$\dim(Z)$-Du Bois and whether pre-$\lfloor \frac{\dim(Z)}{2} \rfloor$-rational $\implies$ pre-$\dim(Z)$-rational; another by Mihnea Popa about seeing what are the possible cohomology tables of the Du Bois complexes. The first question was motivated by trying to extend similar statements which hold in the LCI setting to the non-LCI case while the second question was motivated by trying to see if the various restrictions we currently have on the cohomology tables of the Du Bois complexes (such as Steenbrink vanishing, the sliding rule etc) are the only possible restrictions.

The group tried to address both the questions. First, they tried to prove pre-$\lceil \frac{\dim(Z)}{2} \rceil$-Du Bois $\implies$ pre-$\dim(Z)$-Du Bois for a cone over a smooth projective hypersurface. This entailed trying to play around with certain Bott-type vanishing results for smooth projective hypersurfaces. However, they were unable to prove this statement or produce a counterexample. For the second question, they quickly saw that there is only one Du Bois table, call it T, when $\dim(Z) = 2$ for which it is unclear that there is an example. They spent the rest of the time trying to produce an example of a surface with T as its Du Bois table. They saw that certain types of surfaces (for example, cones over curves) couldn't have T as their Du Bois table, so they instead considered surfaces (for example, $x^2 + y^3 + z^7$) outside these types but got stuck trying to prove that their Du Bois table is T. They plan to continue meeting to discuss this problem.